MECHANICS I
STATICS +++

eBook Version

Published for the Keystone Program
for the 2016-2017 Academic Year

JAMES W. DALLY

ROBERT J. BONENBERGER, JR.

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This book is for Anne
who has shared Jim's dreams for many years.

and to the memory of Ann and Bob
who had encouraged and supported Bob until they were called.

and to higher education which has had such a positive impact on so many lives.

ABOUT THE AUTHORS

James W. Dally (Jim) obtained a Bachelor of Science and a Master of Science degree, both in Mechanical Engineering from the Carnegie Institute of Technology. He obtained a Doctoral degree in mechanics from the Illinois Institute of Technology. He has taught at Cornell University, Illinois Institute of Technology, the U. S. Air Force Academy and served as Dean of Engineering at the University of Rhode Island. He is currently a Glenn L. Martin Institute Professor of Engineering (Emeritus) at the University of Maryland, College Park.

Dr. Dally has also held positions in industry at the Mesta Machine Co., IIT Research Institute and IBM. He is a Fellow of the American Society for Mechanical Engineers, Society for Experimental Mechanics, and the American Academy of Mechanics. He was appointed as an honorary member of the Society for Experimental Mechanics in 1983 and elected to the National Academy of Engineering in 1984. Professor Dally was selected by his peers to receive the Senior Faculty Outstanding Teaching Award in the College of Engineering and the Distinguish Scholar Teacher Award from the University. He was recently awarded the Daniel C Drucker Medal by the ASME and the Archie Higdon Medal by the ASEE.

Professor Dally has co-authored several other books: *Experimental Stress Analysis, Photoelastic Coatings, Instrumentation and Sensors for Engineering Measurements and Process Control, Packaging of Electronic Systems, Mechanical Design of Electronic Systems, Production Engineering and Manufacturing, Experimental Solid Mechanics and Introduction to Engineering Design, Books 1 through 11.* He has authored or coauthored about 200 scientific papers and holds five patents.

Robert J. Bonenberger, Jr. (Bob) obtained B.S.E., M.S. and Ph.D. degrees in Mechanical Engineering from the University of Maryland, Baltimore County and College Park campuses. He has taught undergraduate and graduate students in mechanics, strength of materials, and experimental stress analysis. Currently, he is the Coordinator for the Modern Engineering Materials Instructional Laboratory (MEMIL) and a Keystone Instructor in the Clark School of Engineering at the University of Maryland, College Park.

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PREFACE

In 1998 we began developing a textbook for an integrated course offering for Statics and Mechanics of Materials. After a few years of interactions with several instructors, we published a book titled Design Analysis of Structural Elements that integrated the content of both of these subjects. However, most engineering programs offer these two mechanics subjects in separate courses. To better align the material content with the traditional sequence of course offerings, we have divided its content into the two classical subjects—Mechanics I: Statics and Mechanics II: Mechanics of Materials. In separating the content, we have retained some elements of the integration of the material. The content of the Mechanics I textbook includes methods for determining stresses and strains in uniaxial members, column buckling loads, and a discussion of material properties and material behavior. Because we have included three topics not normally found in traditional Statics books, we have added the three plus signs to the subtitle **Statics +++**.

We introduced the concept of stresses in uniaxial members, because an analysis of the forces in some structural element is incomplete. Determining the force is not sufficient to establish the safety of the structural member or to design its cross section. However, it is easy to introduce stresses in uniaxial members $\sigma = P/A$, and we have taken this important step toward a more complete analysis. Next, we added a chapter on materials and material properties introducing yield and ultimate tensile strength. Determining the stress and comparing this value with the strength of an engineering material enables the student to establish the safety factor or the margin of safety of the structural element. We have found that extending the analysis to incorporate safety and/or design improves the student's interest and motivation.

We often include a project that involves student teams building a model of a truss, which is subsequently tested in the laboratory. The students perform a truss analysis and predict the failure load of their model. In testing the models, we found that compression members on many of the trusses failed at loads much lower than the values predicted by the student teams. These compression members were buckling at relatively low loads, while the stresses were lower than the strength of the model material. This laboratory experience enabled us to discuss elastic instability and to demonstrate buckling. We have added a chapter on Euler (elastic) buckling to this edition to enable the student to study and to begin to understand elastic instability in structures.

The Statics course provides the first exposure of engineering students to the study of mechanics. While Statics is a relatively simple subject, many students find it difficult, and they often perform below expectations. In an effort to improve the curriculum, several members of the faculty at the University of Maryland have been working for the past 18 years to enhance the student's learning experience when studying this first course in mechanics. This textbook reflects many of the changes in the philosophy adopted by the faculty when presenting the subject matter offered in Statics.

After reviewing student performance in Statics in the spring semester of 2016, we were not satisfied with the outcome. To improve course outcomes, we decided to make significant changes in the organization of the textbook. The goal in rearranging many of the chapters and adding and removing content was to place more emphasis on modeling and less on vector manipulation. We found that many students attempted to solve problems without complete and accurate free body diagrams

(FBDs). This leads to solutions with missing terms and the incorrect answers. We noted that students already understood vector manipulations, from their previous studies of vectors in Physics courses; hence, we moved much of the content on vectors, which was previously in Chapter 2, to Appendix E. The new Chapter 2 stresses modeling and shows many examples for preparing FBDs and solving for reaction forces and moments. We also added content on sectioning structural members to visualize internal forces and moments and to provide the FBDs needed to solve for the magnitude of these forces and moments. We trust you will find the new organization of the content helpful in acquiring skills in modelling and in correctly solving the different types of Statics problems you will encounter as you continue to complete your studies in Engineering.

This textbook has a relatively long history. We began developing notes for the first edition of Design Analysis of Structural Elements with a pilot offering in the spring semester of 1999. Since that time, several editions have been published and over 10,000 students have studied Statics and/or Mechanics of Materials using these textbooks. This the fourth edition of **Mechanics I: Statics+++** is another of the eBooks published in this series.

While the errors discovered during the extensive usage have been corrected; errors always occur even with careful proof reading by many diligent people. We would greatly appreciate students and instructors calling any errors to our attention. The e-mail address of one of the authors, Jim Dally, is given on the copyright page.

ACKNOWLEDGEMENTS

As is always the case when major changes are made to the curriculum, College administrators must lead the way. We are indeed fortunate to have several administrators who not only supported this effort, but also insisted that the mechanics offerings be markedly improved. They seek not only a much more favorable educational experience for the students, but also much better understanding and retention of the course content by the students. We thank Deans William Destler, Nariman Farvardin and Darryll Pines who have supported our efforts over the past 18 years.

Special thanks are due to several individuals for their significant contributions to the development of this textbook. Dr. Bill Fourney continues to lead our efforts to change the curriculum providing a more effective approach for presenting Statics and Mechanics of Materials so students would better understand the material and retain this knowledge. Bill has stayed with the project, taught sections of students every semester, and provided the leadership needed to keep others involved and interested. Dr. Hugh Bruck also made major contributions. He also taught an early pilot section and made many excellent suggestions for changes in the sequence of the content. For several years Dr. Robert J. Bonenberger, has assumed the leadership role in organizing the many sections of students taking this course each year.

Many instructors teaching the course made valuable suggestions for improvements to the textbook. These include: Dr. Mary Bowden, Dr. Hugh Bruck, Dr. James Duncan, Dr. Bongtae Han, Dr. Kwan-Nan Yeh, Dr. Peter Sandborn, Dr. Charles Schwartz, Mr. Christopher Baldwin, and Mr. Thomas Beigel of the University of Maryland, College Park.

James W. Dally
College Park, MD

Summer 2016

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LIST OF SYMBOLS

Symbol	Meaning
A	area
%A	percent reduction in area
a	acceleration vector
a, b, c, ….	dimensions
C	constant, center dimension
D	vector difference
D	diameter
d	diameter or distance
%e	percent elongation
E	elastic modulus
E	modulus scale factor
F	force magnitude
F	force as a vector
F_f	friction force
FBD	free body diagram
g	gravitational constant
G	universal gravitational constant
G	shear modulus
h	height
i, j, k	unit vectors
I	moment of inertia
J	polar moment of inertia
k	number, spring rate
L	length dimension
L	load scale factor
ln	natural logarithm
M	moment magnitude
M	moment as a vector
M	multiplier
MA	mechanical advantage
MOS	margin of safety
m	mass, subscript for mode
N	normal force or number
n	number
P	internal force
p	pressure, subscript for prototype
Q	first moment of the area
q	distributed loading
r	radius, radius of gyration, distance
r	position vector
R	reaction force, radius, resistance
R_e	radius of the earth
S	vector sum
S	geometric scale factor
s	distance or dimension
S_{design}	design strength
S_y	yield strength
S_{ys}	yield strength in shear
S_u	ultimate tensile strength
S_{us}	ultimate tensile strength in shear
SF	safety factor
T	torque
t	time
u	unit vector
V	shear force
v	velocity vector
V	volume
W	weight, watt
W	work
w	width dimension
x, y z	Cartesian coordinates
α, β, γ	direction cosines
Δ	delta
δ	deflection or displacement
ε	strain
ε_T	true strain
ϕ	angle of friction, angle of twist
γ	shear strain
π	3.1416 radians
μ	coefficient of friction
ν	Poisson's ratio
Σ	Summation sign
σ	stress
σ_{design}	design stress
σ_f	failure stress
σ_T	true stress
θ	angle
θ_s	angle of repose
τ	shear stress
ω	angular velocity

CHAPTER 1

BASIC CONCEPTS IN MECHANICS

1.1 INTRODUCTION

The subject of mechanics is usually divided into four different subjects, which include:

1. Statics
2. Dynamics
3. Mechanics of Materials
4. Fluid Mechanics

Statics and Dynamics both deal with rigid bodies that are subjected to a system of forces. In the classical study of Statics, we are concerned with determining either internal and/or external forces acting on a structural element that is in a state of equilibrium (usually at rest). In Dynamics, the forces acting on the rigid body produce motion, and the body accelerates or decelerates. The analysis in Dynamics deals with determining position, velocity (angular or linear) and acceleration as some function of time. Newton's laws guide our study of mechanics[1]. Consider Newton's second law:

$$\sum \mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{1.1}$$

where $\Sigma\mathbf{F}$ is the sum of all of the forces acting on the rigid body, m is the mass of the body; $\mathbf{v}$ is the velocity and d/dt is the derivative operator

In dividing the study of mechanics into its four subjects, scholars have considered the special situation when the velocity is constant (often zero) and developed the subject of **Statics** based on this simplification of Newton's second law. In this special situation:

$$\sum \mathbf{F} = 0 \tag{1.2}$$

We study rigid body motion in Dynamics — the velocity of the body is changing ($\mathbf{v} \neq 0$) and the general form of Eq. (1.1) applies. However, in most situations the mass of the rigid body is constant (dm/dt = 0) and if this is the case, Eq. (1.1) reduces to:

$$\sum \mathbf{F} = m\frac{\mathbf{dv}}{\mathbf{dt}} = m\mathbf{a} \tag{1.3}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the acceleration of the rigid body.

[1] Sir Isaac Newton (1642-1727) formulated three laws of motion and the law of universal gravitational attraction.

Basic Concepts in Mechanics

In the study of both Statics and Dynamics, the material from which the body is manufactured is of no concern providing the body remains essentially rigid under the action of the imposed forces. However, in studying Mechanics of Materials[2], the deformation of the body is an essential consideration in the analysis. With this approach we assume that the deformations of the body are small and that plane sections remain plane after the body deforms. These assumptions enable us to determine the distribution of internal forces and stresses in the body. The material from which the body is fabricated is of critical importance in Mechanics of Materials for two reasons. First, the deformations of the body due to the forces are markedly affected by the rigidity of the material (its elastic modulus). Second, the behavior of the body depends on its strength. Whether the body fails or not depends on its strength, which is a physical property.

In fluid mechanics the situation is entirely different. The body is either a gas or a liquid or a two phase mixture where both liquid and gasses are present. The deformations are so large they are considered to be flows. The flow can be compressible (gasses under higher pressures) or incompressible (liquids or gasses at low pressures). The flow can occur in closed channels or open channels. It may be internal to a conduit or external to some surface. The flow may be stable (laminar) or unstable (turbulent). The phase of the material may change during a process and the resulting flow consists of two-phases (some liquid and some gas). Because of the complexities inherent in fluid mechanics, it is studied after a student has established a thorough understanding of the other three branches of mechanics.

Before beginning the design analysis of any component, vehicle, structure or even a model of a structure, it is essential to thoroughly understand several basic concepts in mechanics and the behavior of materials. The first two principal subject areas in these fields are **Statics** and **Mechanics of Materials**. In this chapter, we introduce many of the basic concepts and physical laws included in these two subjects. These concepts and/or laws provide a foundation upon which the design analysis of machine components and structures is based. We also briefly describe the history of mechanics to give you a sense of the age of the principles used in analyzing modern machine elements and structural components.

1.2 STATICS AND MECHANICS OF MATERIALS

There are two closely related courses in mechanics, namely **Statics** and **Mechanics of Materials**. In studying **Statics**, we assume the body under consideration is perfectly rigid. As such, it does not deform under the action of applied forces. You will solve many different types of problems determining **external forces** acting on some structures and the **internal forces** developed in others by using only **free body diagrams (FBDs)** and the **equilibrium equations**. The equilibrium equations are developed from the laws of motion proposed by Sir Isaac Newton to describe the balance of forces acting on a stationary body or on a body moving with a constant velocity. Solutions for forces acting on bodies in equilibrium employ only three basic steps:

- Construct a complete set of free body diagrams[3].
- Apply the appropriate equations of equilibrium.
- Execute the mathematics required to solve one or more equations of equilibrium.

Statics is a relatively easy subject that you can quickly master. You will find it easy to solve many different problems with the procedure outlined above.

Mechanics of Materials is a relatively simple extension of Statics, which considers the effect that **material deformations**[4] have on the **internal stresses** generated in a body by a system of applied

---

[2] The course Mechanics of Materials is also known as Mechanics of Deformable Bodies or Strength of Materials.

[3] A complete set of FBDs is essential in successfully executing solutions to problems in Mechanics.

[4] In Mechanics of Materials, we assume the deformations are so small that they do not affect the magnitude or the direction of the internal and external forces acting on the body.

external forces. To solve the problems that arise in Mechanics of Materials, we begin with the same three steps, and then add two more. The additional steps are to accommodate the effect of the deformations when the body is subjected to a system of external forces:

- Construct a complete set of free body diagrams.
- Apply the appropriate equations of equilibrium.
- Assume the geometry of the deformations (usually plane sections remain plane).
- Employ the appropriate relations between stress and strain.
- Execute the mathematics required in solving the equations.

By comparing these lists, it is evident that Statics and Mechanics of Materials are closely related. Indeed, we must use the equilibrium equations in Statics before we can begin to solve typical problems in Mechanics of Materials. However, the solutions to Statics problems are obtained on a global scale, whereas the solutions to Mechanics of Materials problems are obtained on a local scale.

In studying Mechanics of Materials, we quickly encounter the concepts of stress and strain. Mathematically these concepts are somewhat complex because they are **tensor** quantities, but physically they are simple and easily understood. Stress is a concept based on the equilibrium of a portion of a body. Consider a part of a body produced by sectioning. The internal force acting on the section cut is developed by some distribution of stresses over the area exposed by the cut.

Strain, on the other hand, is a geometric concept. We determine strain by the change in geometry that occurs when a body deforms under the action of a system of external forces. When considered individually, both stress and strain are independent of the material from which the body is fabricated. It is only when we write a relation for strain in terms of stress, or vice versa, that we must consider the properties of the material used to fabricate the body.

Materials used in constructing various structures are largely ignored in the study of Statics. The equilibrium equations are the same for all materials, and the internal and external forces for statically determinant structures[5] do not depend upon the materials employed to fabricate the structure. Materials are much more important in studies of the Mechanics of Materials. We usually relate stress and strain in our solutions, and must employ the elastic constants that describe the rigidity of the materials. In addition, we may be required to predict the margin of safety for a structure subjected to specific loading. In the solution to problems dealing with structural failure, we employ the appropriate **"strength"** of the material.

In the two textbooks used in Mechanics I and Mechanics II, we stress the physical aspects of both Statics and Mechanics of Materials. We consider it essential that you construct complete free body diagrams to model the structure and to define the unknown forces. We are more interested in your understanding of the equations of equilibrium than your use of **vector algebra** to solve them. Of course, it is important that you be able to correctly manipulate the equations resulting from the application of the principle of equilibrium. Whether these equations are written using trigonometric functions or in vector format is of lesser concern providing they are correct.

The general subject of Mechanics of Materials is both interesting and important. Let's try to enjoy the experience of learning this fundamental subject. In writing these textbooks, we have used many examples to:

1. Illustrate drawing free body diagrams.
2. Show the correct procedures for applying the equilibrium equations.

[5] Structures are classified as statically determinant if the reactions at their supports may be determined using the equations of equilibrium.

3. Show the geometric changes due to deformations of the body under load.
4. Illustrate distributions of stress over areas exposed by section cuts.
5. Demonstrate the procedures and techniques involved in solving different types of problems.

1.3 HISTORY OF MECHANICS

As a student of the first course in mechanics, you are probably in your late teens or early twenties. It might be hard for you to imagine that mechanics has been under development for more than 2,000 years. Early pioneers of modern mechanics, about two or three hundred years ago, understood most of the material included in this book. Although research is still conducted in mechanics, this first course in Statics covers the classical content that is considered the foundation for mechanics. To illustrate that mechanics is a classical subject, let's note the contributions of a few of the pioneers, who established the foundations of mechanics. For a much more extended treatment of the history of mechanics see references [1, 2].

One of the earliest contributors to our knowledge of mechanics was Archimedes (287 – 212 BC) who discovered that we could use a lever to increase the weight that a person could lift. The concept of the lever with its fulcrum used in lifting a weight W, with an external force W/4, is illustrated in Fig. 1.1. Archimedes also described the use of pulleys and inclined planes for moving materials at construction sites. On a different application of engineering in ancient times, Archimedes is credited with discovering the law of buoyancy, where the upward force on a submerged body is equal to the weight of the water displaced by that body.

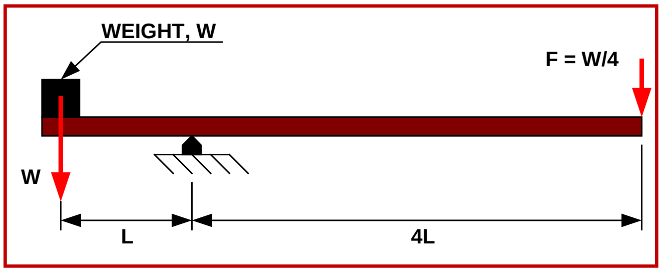


Fig. 1.1 The lever with a fulcrum permits a small applied force to be increased by the lever ratio.

Leonardo da Vinci (1452 – 1519) was a man with incredible talent and one of the great Renaissance masters. He was a painter, sculptor, musician, architect and engineer. At the age of about 30, he served as the principal engineer for the Duke of Milan supervising the construction of bridges and war machines for the Duke's military ventures [3]. Leonardo is not known for his mathematical discoveries, but rather for his engineering innovations that implied his thorough understanding of mechanics.

The 17th century was the height of the Renaissance period for science and probably the most productive for developing the foundations of mechanics. During this century, Robert Hooke (1635 – 1703) introduced the concept of elasticity by noting that a body deformed in proportion to the applied forces. This concept leads to the well-known Hooke's law:

$$\sigma = E\,\varepsilon \qquad (1.4)$$

where σ is the stress, ε is the strain, and E is the material property known as the modulus of elasticity.

The linearity of Hooke's law is illustrated in Fig. 1.2. Robert Hooke also developed springs and used them in watches and clocks to improve their accuracy and to store the energy required to drive the clock's mechanisms. He discussed planetary motion, but did not develop the mathematics necessary to describe the observed motion of the planets. It was Sir Isaac Newton, who extended Hooke's early ideas about planetary motion, and wrote the gravitational law that predicts the motion of the planets.

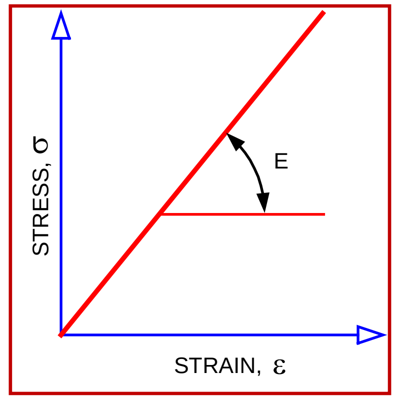


Fig. 1.2 The linear elastic relation between stress and strain.

Sir Isaac Newton (1642 – 1727) was a giant in both mathematics and mechanics. He was only 24 when he developed the mathematics known today as differential calculus. Later Newton turned his attention to planetary motion, and formulated his famous laws of motion. We will cover these three laws in more detail in the next section. However, these laws are the foundation of much of the content contained in mechanics courses on Statics and Dynamics. He is best known for his universal law of gravity [4] that explains the attractive forces between two bodies. The gravitational force is an internal force because it acts on each elemental volume of mass within a body. The attractive forces **F** between two masses, illustrated in Fig. 1.3, are given by:

$$F = G\, m_a\, m_b / r^2 \tag{1.5}$$

where F is the magnitude of the gravitational force.

$G = 6.673 \times 10^{-11}$ m^3/(kg - s^2) is the universal gravitational constant.

m_a and m_b are the masses of bodies A and B.

r is the distance between the centers of the two bodies.

Because Newton's law of gravity is extremely important, it will be described in more detail later in this chapter.

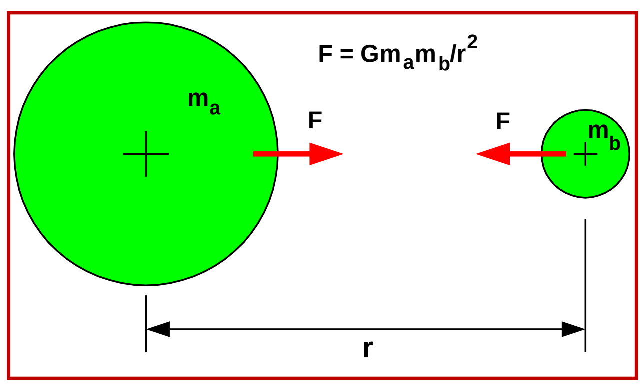


Fig. 1.3 Illustration of the attractive forces on both bodies developed by gravity.

Jacob Bernoulli[6] (1654 – 1705) made two contributions to Mechanics of Materials; one was correct and the other was in error. We cite his contributions to indicate that even the great and famous make mistakes from time to time. Jacob was interested in determining the elastic curve representing the deflection of a

---

[6] There were three famous Bernoulli's —Jacob, John and Daniel. John, the younger brother of Jacob, was an outstanding mathematician. Daniel, the son of John, was famous because of his work in fluid mechanics, his outstanding book Hydrodynamica, and for his suggestion to Euler for deriving the equations for elastic curves.

beam. He correctly showed that the curvature of the beam was proportional to the bending moment at each point along the length of the beam. He then made the error of assuming that the elastic curve was positioned along the bottom edge of the beam. We will establish in our studies of Mechanics of Materials that the elastic curve must coincide with the neutral axis of the beam, which does not coincide with the bottom surface of the beam.

The history of mechanics is rich with accomplishments of many mathematicians. As the subject evolved, engineers designing safe, efficient structures and machines such as airplanes, automobiles, bridges, machines, skyscrapers, and space satellites reduced the mathematical formulations presented in this textbook to practice.

1.4 NEWTON'S LAWS OF MOTION

Sir Isaac Newton wrote three laws of motion that form the foundation for both Statics and Dynamics. These laws are:

1. If the sum of all of the forces acting on a body is zero (i. e. $\Sigma \mathbf{F} = 0$), the body will:
    - Remain at rest.
    - Move at a constant velocity along a straight line.

2. If the sum of all of the forces **F** acting on a body is not zero, the body will undergo a time rate of change of the linear momentum (m**v**) given by:

$$\sum \mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \qquad (1.1)$$

When the mass m of the body remains constant with respect to time dm/dt = 0, and Eq. (1.1) reduces to:

$$\sum \mathbf{F} = m\frac{\mathbf{dv}}{\mathbf{dt}} = m\mathbf{a} \qquad (1.3)$$

3. The force exerted by body A on body B is equal in magnitude but opposite in direction to the force that body B exerts on body A. This third law is often called the law of action and reaction.

1.4.1 Newton's First Law

The first of Newton's laws is written as:

$$\Sigma \mathbf{F} = 0 \qquad (1.2)$$

Another way of representing Eq. (1.2) is:

$$\mathbf{F_1} + \mathbf{F_2} + \ldots\ldots + \mathbf{F_n} = 0 \qquad (1.2a)$$

where a system of n external forces is acting on the body.

Equation (1.2), in one form or the other, is used extensively in both Statics and Mechanics of Materials. It is used every time that we write the **equilibrium** equations. To illustrate the meaning of the mathematical symbol **ΣF**, let's examine the drawing of the body presented in Fig. 1.4. We have four forces acting on this body. They are pointed in four different directions and they have four different magnitudes. The forces are **vector** quantities that must be characterized by specifying both a magnitude and a direction[6]. When the **vector sum** of these forces is zero, the body is in equilibrium.

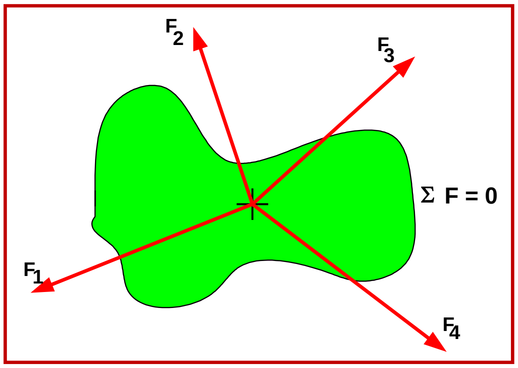


Fig. 1.4 Four forces acting on a body that is in equilibrium if $\Sigma\mathbf{F} = 0$.

We may recast the vector representation of Eq. (1.2) by writing the equivalent **scalar equations** as:

$$\Sigma F_x = 0 \qquad \text{and} \qquad \Sigma F_y = 0 \qquad (1.2b)$$

$$F_{1x} + F_{2x} + \ldots + F_{nx} = 0 \qquad \text{and} \qquad F_{1y} + F_{2y} + \ldots + F_{ny} = 0 \qquad (1.2c)$$

We have accounted for the direction of the forces in this equation by considering only those forces in either the x or y directions. When the forces are constrained to the x and y directions, we employ the **scalar form of the equilibrium equations**. However, when the forces act in directions other than the x and y, the vectors must first be **decomposed** into their **components** acting in the x and y direction before they can be used in the scalar equilibrium equations.

Let's consider an example to demonstrate the application of the equilibrium equation. We will first consider only forces acting in either the x or y directions.

EXAMPLE 1.1

Consider the forces acting on the body shown in Fig. E1.1, and determine if it is in equilibrium.

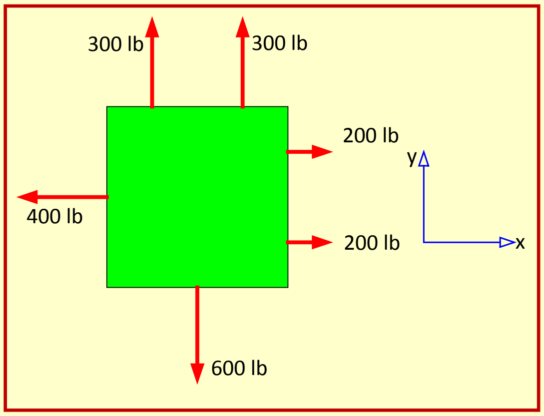


Fig. E1.1

[6] We use bold font to represent forces as vector quantities when both magnitude and direction are specified. When we consider only the magnitude of a force, normal fonts are used to represent its scalar magnitude.

Solution:

We have inserted the x-y coordinate system in Fig. E1.1, and employ it as a reference when summing the external forces acting on the square block. We apply Eq. (1.2) to determine if this block is in equilibrium. When we write $\Sigma \mathbf{F} = 0$, it is important to note that **F** is a vector quantity and the summation of the forces is the **vector form of the equilibrium equation**. Let's take this fact into account by considering first all of the forces in the x direction, and then all of the forces in the y direction. With this approach, we are accounting for both the directions and the magnitude of each of the six forces acting on the block. We write:

$$\Sigma F_x = 0 \qquad \text{and} \qquad \Sigma F_y = 0$$

$$+200 + 200 - 400 = 0 \qquad \text{and} \qquad +300 + 300 - 600 = 0$$

These results show that the equilibrium conditions are satisfied in both the x and y directions. The forces in the positive directions of both x and y cancel with those in the negative directions. Hence, the body is in equilibrium.

1.4.2 Newton's Second and Third Laws

Newton's second law $\sum \mathbf{F} = \frac{d}{dt}(m\,\mathbf{v}) = m\,\mathbf{a}$ is the equation used most frequently in the study of Dynamics. When the sum of the forces is not zero, the body of mass m is subjected to an acceleration **a**. Depending on your choice of engineering disciplines, you may study Newton's second law in detail later in the curriculum.

The third law is often called the law of action and reaction. We illustrate the concept of active and reactive forces in Fig. 1.5. In this illustration, a spherical shaped mass m with a weight W rests on the floor at a contact point. The sphere is in equilibrium under the action of two forces. The first force is the weight W due to gravity that acts downward. The second is the reaction force developed at the contact point. The reaction force R is equal in magnitude to the weight W, but opposite in direction.

When any two bodies are in contact (i. e. the sphere and the floor), two forces develop at the contact point. These forces are equal in magnitude and opposite in direction.

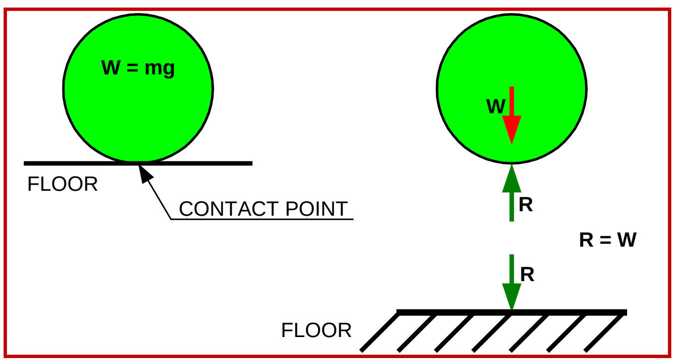


Fig. 1.5 The active force W, due to gravity, produces a reaction force R at the contact point.

EXAMPLE 1.2

Suppose that a Corvette Stingray is parked on a stretch of pavement, as shown in Fig. E1.2. Let's determine the reaction forces that develop between the pavement and the tires.

Solution:

We start by drawing a free body diagram (FBD) of the automobile. In the free body diagram shown in Fig. E 1.2, the automobile is separated from the pavement. The auto is suspended in space and maintained in equilibrium by vertical (y direction) reaction forces R_L and R_R applied to the tires.

At the two contact points with the pavement, we have drawn arrows downward representing the equal and opposite contact forces R_L and R_R that are applied to the pavement. The reaction forces R_L and R_R are drawn upward on each tire. Clearly the applied force W and the sum of the reaction forces $R_L + R_R$ are equal in magnitude and opposite in direction. Is the Stingray in equilibrium? Why is R_R larger than R_L?

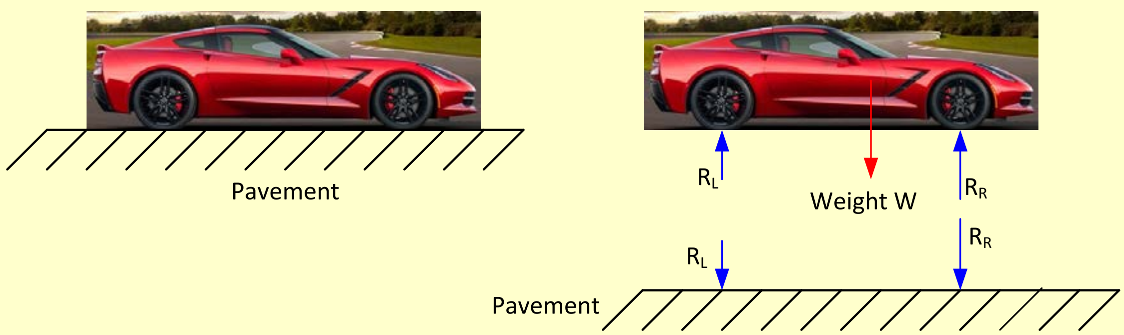


Fig. E 1.2 Free body diagram of a Stingray parked on a stretch of pavement.

1.5 FORCES

Statics involves a study of forces that act on and within members of a structure. We seek to determine the external forces acting on bodies, and the internal forces developed by stresses within structural members. External forces occurring under static (steady state) conditions include:

- Gravitational
- Pressure acting over a defined area
- Friction
- Magnetic
- Electrostatic

In addition, forces developed under dynamic conditions are referred to as inertial forces and include:

- Centrifugal
- Centripetal
- Coriolis

First, let's examine the forces due to gravity, because they are by far the most important. We continuously work and expend huge amounts of energy to overcome gravitational forces. Gravitational forces are the primary concern when we design buildings and bridge structures against failure by collapse or rupture. Even in vehicle design, where other dynamic forces are significant, gravitational forces are critical in the design of both the structure and the power train.

Weight is a force produced by the Earth's gravitational pull on the mass of our body as illustrated in Fig. 1.6. Suppose we examine the force on a body due to gravity by modifying Eq. (1.5) and letting:

$m_a = m_e$ the mass of the Earth.
$m_b = m_b$ the mass of our body.
$r = R_e$ the radius of the Earth.

Then we rewrite Eq. (1.5) to give:

$$F = \frac{Gm_e m_b}{R_e^2} \tag{1.6}$$

In setting $r = R_e$, we assumed that Earth bound bodies, either those of people or objects, are very small compared to the radius of the Earth, which is 3,960 mi. or 6.37 x 10^6 m.

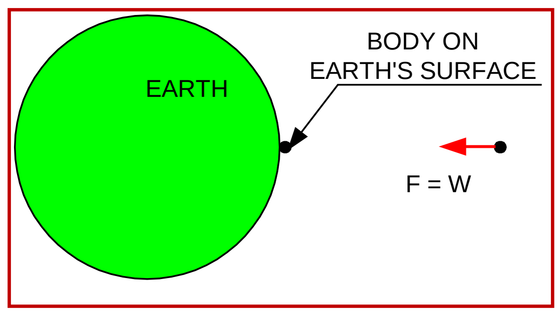


Fig. 1.6 Bodies on Earth's surface are small relative to the Earth's radius.

Next collect together all of the quantities in Eq. (1.6) that characterize the Earth, and set them equal to g_e.

$$g_e = \frac{Gm_e}{R_e^2} \tag{1.7}$$

Note that g_e is the **gravitational constant** equal to 32.17 ft/s^2 or 9.807 m/s^2. We will drop the subscript in subsequent discussion of the gravitational constant with the understanding that g is to be applied to Earth bound bodies.

Strictly speaking g is not constant because it varies a small amount as we move from one location on the Earth to another. The Earth is not a perfect sphere, and R_e does not remain constant, as we move from Pikes' Peak to Death Valley. However, the variations are so small that we neglect them without introducing significant error in our design analyses.

Combining Eqs. (1.6) and (1.7) gives:

$$F = m_b\, g = W \tag{1.8}$$

The force F in Eq. (1.8) is the weight W of a body on Earth having a mass m_b.

From our definition, it clear that the units of g are ft/s^2 or m/s^2; hence, g is an acceleration. Indeed, if we jump from a diving platform, our body accelerates with a = g until we hit the water. Clearly, this relationship is consistent with Newton's second law.

If we travel from Washington, D. C. to Denver, CO, we observe that our weight remains essentially the same. So we become confused and believe that the constant in Eq. (1.8) is our weight. It is a reasonable thought, but erroneous. When measuring weight, it is essentially constant, if we remain Earth bound. However, the constant quantity in Eq. (1.8) is not the weight W, but the mass m_b. To prove this statement, go to the moon, and measure your weight. It is known that we weigh much less on the moon — about one sixth as much as here on Earth. Because our mass m_b is constant, we weigh less because the gravitational constant for the moon is only about g/6. The smaller gravitational constant for the moon is due to its much smaller mass and radius when compared to the Earth. The mass of our body is the same whether we are on the moon, Mars, the Earth, or anywhere in space.

EXAMPLE 1.3

(a) Determine the weight of an object with a mass of 31 kg (the kilogram is the unit for mass in the International System).

Solution:

We use Eq. (1.8) to calculate the weight as:

$$F = W = mg = (31\ kg)(9.807\ m/s^2) = 304.0\ N$$

where N is the symbol for **newton**, which is the unit for force in the SI system. One newton is also equivalent to 1 $kg\text{-}m/s^2$. We will discuss units used in the SI system in more detail in Section 1.9.

(b) Determine the weight of an object with a mass of 9.25 slugs (a slug is the unit for mass in the U.S. Customary system).

Solution:

Using Eq. (1.8) again gives:

$$F = W = mg = (9.25\ slug)(32.17\ ft/s^2) = 297.6\ lb$$

The slug has units of $(lb\text{-}s^2)/ft$. An object weighing 32.17 lb on Earth has a mass of 1 slug.

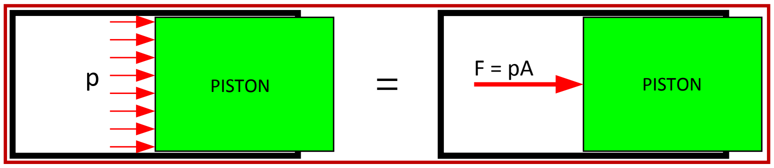
Did you drive your car to the university today? If so, the pressure developed by the combustion of gasoline-oxygen vapor within the cylinders of your car's engine provided the force to propel it along the roads. Pressure, p acts over an area, A of some surface to create a force.

$$F = p\,A \tag{1.9}$$

where A is the cross sectional area of the piston.

We illustrate the force F produced by the action of the pressure p on the piston shown in Fig 1.7. The magnitude of the force is determined by using Eq. (1.9); its direction is normal to the surface of the piston.

Fig. 1.7 The pressure acting on the piston produces a force F = pA.



1.6 EXTERNAL AND INTERNAL FORCES

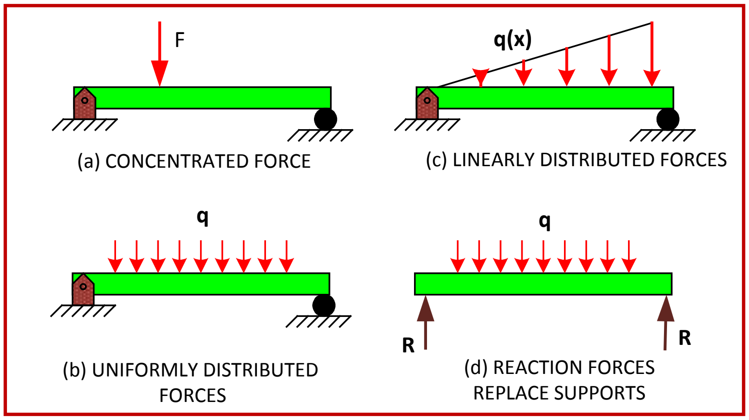
In dealing with forces, we distinguish between those that are applied to the structure **(external)**, and those that develop within a structural element **(internal)**. The external forces include the active loads applied to the structure, such as those shown in Fig. 1.8.a-c, and the reaction forces, shown in Fig. 1.8d, that develop at the supports to maintain the structure in equilibrium.

In Fig. 1.8a, a simply supported beam is loaded with a **concentrated force** at a local point near its center. A concentrated force, applied at a point, is an idealization. Forces are always distributed over some area; however, with concentrated forces, we assume that the area is so small that it approaches a point. The symbol F is used to designate the magnitude of concentrated forces. The arrow indicates its direction.

In Fig. 1.8b, the beam is loaded with **uniformly distributed forces** that are applied over most of its length. Uniformly distributed forces along beams are specified in terms of force/unit length (i.e. lb/ft or N/m). The symbol q is used to designate the magnitude of the distributed forces applied to a beam.

Distributed forces that are increasing, as we move from its left end of the beam to its right, are illustrated in Fig. 1.8c. Again, the symbol q is used to designate the magnitude of the distributed forces; however, in this case we must recognize that q is a function of position x along the length of the beam, designated by q(x).

Fig. 1.8 Examples of different types of external forces applied to the structure (beam).



The last example of external forces is shown in Fig. 1.8d. The beam, with its uniformly distributed forces, is identical with that shown in Fig. 1.8b, but its supports have been removed. The reaction forces developed by the supports to maintain the beam in equilibrium are shown as concentrated forces. The symbol R will be used to designate the magnitude of reaction forces.

Internal forces develop within a structural member due to the action of the applied external forces. These internal forces are not visible, although we try to visualize them by making imaginary section cuts through a structural member. Let's examine a cylindrical bar subjected to external forces F applied at each of its ends, as shown in Fig. 1.9. We make a section cut perpendicular to the axis in the central region of the bar. This is an imaginary cut, not a real one, but it permits us to visualize the two segments of the bar. We examine the segment on the left, and find the **normal stresses** σ, which are uniformly distributed over the area exposed by the section cut. When this stress is integrated over the area of the bar, an **internal force** P_{int} is generated that acts along the axis of the bar. The magnitude of P_{int} is given by:

$$P_{int} = \int \sigma \, dA \qquad (1.10)$$

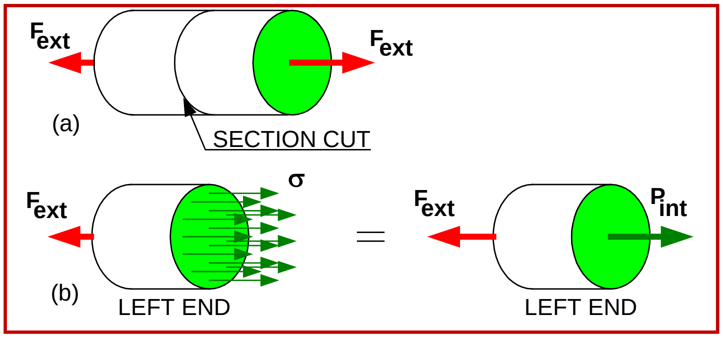


Fig. 1.9 (a) A tension bar with external forces and a section cut dividing the bar into two parts.
(b) Two different representations of the left end of the bar.

The left end of the bar must be in equilibrium, which implies that:

$$\Sigma F_x = 0 \qquad (1.2)$$

Summing the forces in the x direction gives:

$$P_{int} = F_{ext} \qquad (1.11)$$

In this elementary example of a rod in tension, we have found the relation between the internal force P within the bar and the external forces F_{ext} applied at its ends. We will use the same approach throughout this text in solving much more complex problems. In your solutions to the assigned exercises, remember to:

1. Draw complete and accurate FBDs.
2. Make an appropriate section cut or cuts.
3. Draw FBDs showing external and internal forces including all reaction forces.
4. Use the equations of equilibrium.
5. Solve for internal forces in structural members.

1.7 MOMENTS

If you have ever used a wrench or screwdriver to tighten a bolt or screw, you have generated a moment. A moment M_o about a point O is produced when a force F, as shown in Fig. 1.10, is applied in such a manner that it tends to cause a body to rotate about this point. The magnitude of a moment produced by a force is dependent on the location of point O, and is given by:

$$M_o = F\, d \qquad (1.12)$$

where d is the perpendicular distance from the point O to the line of action of the force F.

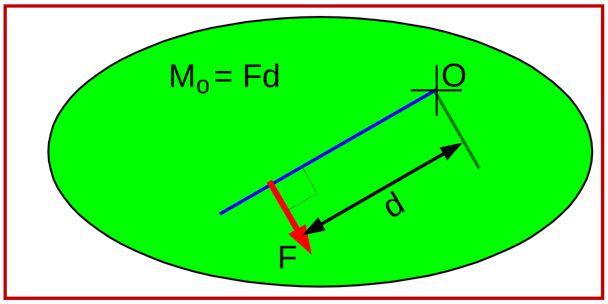


Fig. 1.10 A moment produced by a force depends on the position of point O.

The units of a moment $\mathbf{M_o}$ are given as N-m or ft-lb. $\mathbf{M_o}$ is a vector quantity; hence, it must be specified with both magnitude and direction. Its magnitude is given by Eq. 1.12, and the direction of the vector $\mathbf{M_o}$ is perpendicular to the plane in which both F and d lie. As shown in Fig. 1.11, the moment $\mathbf{M_o}$ has a sense of direction. In this illustration, the moment $\mathbf{M_o}$, when viewed from above, tends to rotate the body in a **counterclockwise** direction and is **positive**.

To determine the sign of the moment, we use the **right hand rule**. In applying this rule, place the palm of your right hand along the axis of rotation and point your fingers in the direction of the force and rotate your hand. If the direction of the force causes you to rotate **counterclockwise**, with your thumbs pointing up from the plane in which both F and d lie, then the moment is **positive**. However, if you must rotate your hand **clockwise**, with your thumb pointing downward, the moment is **negative**.

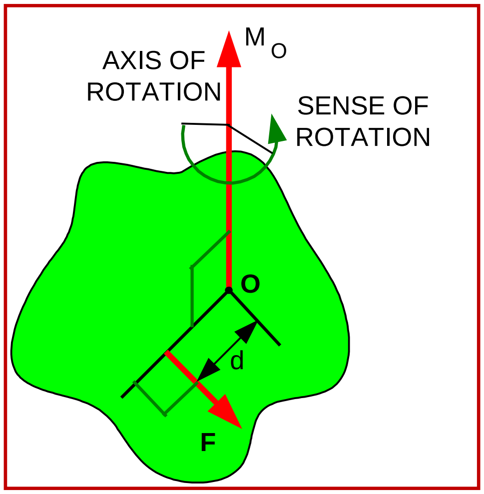


Fig. 1.11 A graphic illustration of the moment $\mathbf{M_o}$ as a positive vector quantity.

EXAMPLE 1.4

A hexagonal headed bolt is tightened with a wrench, as shown in Fig. E1.4. A 32-lb force is applied to the handle of the wrench to produce a moment (torque). If the distance from the centerline of the bolt to the point of application of the force is d = 9 in., find the applied moment (torque).

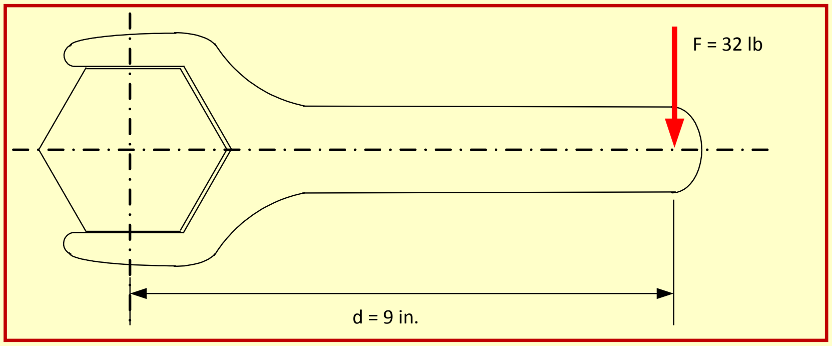


Fig. E1.4

Solution:

From Eq. (1.12), we write

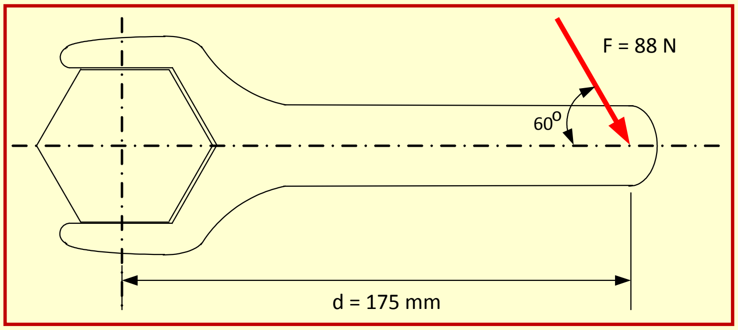
$$M = F\,d = (32)(9) = 288 \text{ in-lb} = 24.0 \text{ ft-lb} \qquad \text{(a)}$$

We consider this moment to be **negative** because it tends to produce a **clockwise** rotation about the head of the bolt.

EXAMPLE 1.5

Suppose a force of 88 N is applied to the wrench, as shown in Fig. E1.5. Determine the torque (moment) applied to the hex headed bolt.

Fig. E1.5



Solution:

Let's resolve the 88 N force into two components — one perpendicular to the axis of the wrench and the other parallel.

The component of force parallel to the axis of the wrench is obtained by:

$$F_{=} = (88)\cos(60°) = 44.0 \text{ N} \qquad \text{(a)}$$

Note that $F_{=}$ does not produce a moment (torque) on the bolt because its moment arm d = 0. The component perpendicular to the axis of the wrench is given by:

$$F_{\perp} = (88)\sin(60°) = 76.21 \text{ N} \qquad \text{(b)}$$

To determine the moment that this component of force produces on the head of the bolt, we use Eq. 1.12 and write:

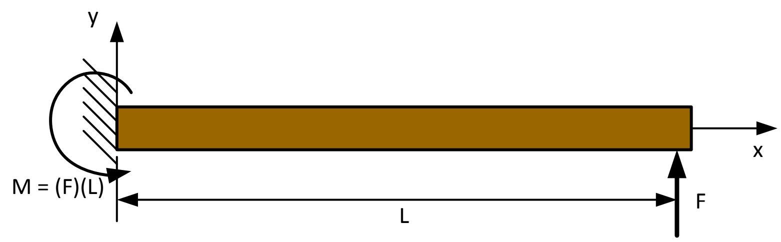
$$M = F\,d = (76.21)(0.175) = 13.34 \text{ N-m} \qquad \text{(c)}$$

Again, this moment is **negative** because it tends to produce a **clockwise** rotation about the head of the bolt.

1.8 EXTERNAL AND INTERNAL MOMENTS

Moments like forces are classified as either external or internal. The wrench in EXAMPLE 1.4 applied an external moment (often called torque) to the head of the bolt. Consider the sketch of a cantilever beam with a length L, as shown in Fig. 1.12. The beam is subjected to a force F applied at its free end and directed upward. This is an external force that creates an external moment M at the beam's fixed end that moment is equal to M = F ×L.

Fig. 1.12 A cantilever beam with a force F applied at is free end.



If we make an imaginary section cut at the fixed end of the beam, we expose the bending stresses σ produced by the moment M, as shown in Fig. 1.13.

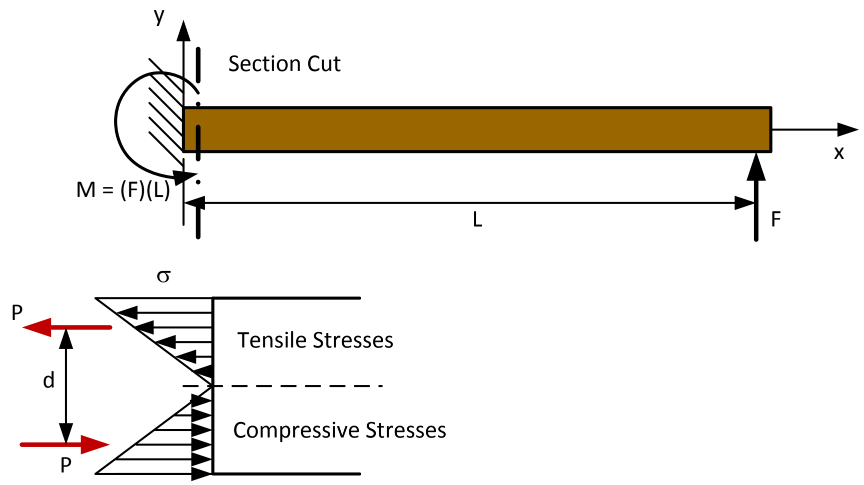


Fig. 1.13 A section cut reveals the bending stresses at the fixed end of the cantilever beam.

The stresses on the top half of the beam are tensile and those on the bottom half of the beam are compressive. When the tensile stresses are integrated over the area in the top half of the beam, the result is a tensile force P in the negative x direction. When the compressive stresses are integrated over the area in the bottom half of the beam, the result is a compressive force in the positive x direction. These two equal and opposite forces P, give a couple equivalent to the moment M. The distance between the two P forces is d. Hence:

$$M_{Ext} = F \times L = M_{Int} = P \times d \qquad (1.13)$$

1.9 BASIC QUANTITIES AND UNITS

In the study of mechanics, we encounter four basic quantities — length, time, force and mass. These quantities are shown with their respective units for the SI and the U. S. Customary systems of units in Table 1.1.

Table 1.1
Basic Quantities and Units

System of Units	Length	Time	Mass	Force	g
International System of Units (SI)	meter (m)	second (s)	kilogram (kg)	newton (N), (kg-m)/s^2	9.807 m/s^2
U. S. Customary (FPS)	foot (ft)	second (s)	slug (lb-s^2)/ft	pound (lb)	32.17 ft/s^2

The basic units are not independent because Eq. (1.3) requires that the units be **dimensionally homogenous**. To maintain the dimension homogeneity of Eq. (1.3), we define the units for length, time and mass in the **SI** system, and then derive the remaining basic unit for the force, the **newton**, in terms of those units. For the **U. S. Customary** system, the basic unit for mass, the **slug**, is derived in terms of the units for length, time, and force.

In the International System of Units (SI), the length is given in meters (m), the time in seconds (s), and the mass in kilograms (kg). The unit for force is called a newton (N) in honor of Sir Isaac. The newton is derived from Eq. (1.3) so that a force of 1 N will impart an acceleration of 1 m/s^2 to a mass of 1 kg [i. e. 1 N = (1 kg) (1 m/s^2)]. For dimension homogeneity, it is clear that N is equivalent to (kg-m)/s^2. In the SI system the gravitation constant g = 9.807 m/s^2. With this value of the acceleration due to gravity on Earth, the weight of a mass of 1 kg is:

$$W = mg = (1\text{kg})(9.807\ \text{m/s}^2) = 9.807\ \text{N}$$

In the U. S. Customary System, the length is given in feet (ft), force in pounds (lb), and time in seconds (s). The unit for mass is called a slug, which is derived from Eq. (1.3) so that a force of 1 lb will impart an acceleration of 1 ft/s^2 to a mass of 1 slug [i. e. 1 lb = (1 slug)(1 ft/s^2)]. For dimension homogeneity, it is clear that a slug is equivalent to (lb-s^2)/ft. In the U. S. Customary System the gravitation constant g = 32.17 ft/s^2. With this value of the acceleration due to gravity on Earth, the mass of a body weighing 32.17 lb is:

$$m = W/g = 32.17\ \text{lb}/(32.17\ \text{ft/s}^2) = 1\ \text{slug}$$

In addition to the quantities presented in Table 1.1, we deal with several other quantities in our studies of Statics and Mechanics of Materials. These quantities are listed in Table 1.2

Table 1.2
Units of Other Frequently Used Quantities

System of Units	Moment M	Stress σ	Strain ε
SI, (Nms)	newton-meter, (N-m)	pascal, (Pa) = (N/m^2) mega pascal, (MPa) = (N/mm^2)	dimensionless
U. S. Customary, (FPs)	foot-pound, (ft-lb)	pound/square foot, (lb/ft^2) pound/square in., (lb/in.^{2}) = psi	dimensionless

We have shown the quantities in Table 1.2 in terms of the basic units for length, time and force. However, in practice other units are often employed. For instance, a moment may be expressed as in-lb instead of ft-lb, and a stress expressed as MPa instead of Pa or psi (lb/in^2) instead of (lb/ft^2). We are fortunate that strain is a dimensionless quantity; we will not have to convert from one system of units to another, or to convert from one unit to another within the same system. A technique to convert from one system of units to another is described in the next section.

Often the metric prefixes are employed in expressing numerical results. Indeed, we used a metric prefix M with the unit for stress (Pa) in the previous paragraph. The metric prefixes are useful in dealing with either very large or very small numbers. Accordingly, it is common practice to employ them together with the symbol for the units (e.g. MPa).

Table 1.3
SI Prefixes

Multiplication Factor	Prefix Name	Prefix Symbol
10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto*	h
10^{1}	deka*	da
10^{-1}	deci*	d
10^{-2}	centi*	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

*To be avoided when possible.

1.10 CONVERSION OF UNITS

We show two different techniques for the conversion of units from one system to another. The first technique utilizes the conversion factors listed in Table 1.4.

Table 1.4
Unit Conversion Factors

Quantity	U. S. Customary	SI Equivalent
Acceleration	ft/s^2 in/s^2	0.3048 m/s^2 0.0254 m/s^2
Area	ft^2 in^2	0.0929 m^2 645.2 mm^2
Distributed Load	lb/ft lb/in.	14.59 N/m 0.1751 N/mm
Energy	ft-lb	1.356J
Force	kip = 1000 lb lb	4.448 kN 4.448N

Quantity	U. S. Customary	SI Equivalent
Impulse	lb-s	4.448 N-s
Length	ft in mi	0.3048 m 25.40 mm 1.609 km
Mass	lb mass slug ton mass	0.4536 kg 14.59 kg 907.2 kg
Moments or Torque	ft-lb in-lb	1.356 N-m 0.1130 N-m
Area Moment of Inertia	in^4	0.4162×10^6 mm^4
Power	ft-lb/s hp	1.356 W 745.7 W
Stress and Pressure	lb/ft^2 lb/in^2 (psi) ksi = 1000 psi	47.88 Pa 6.895 kPa 6.895 MPa
Velocity	ft/s in/s mi/h (mph)	0.3048 m/s 0.0254 m/s 0.4470 m/s
Volume	ft^3 in^3 gal	$0.02832\ m^3$ $16.39\ cm^3$ 3.785 L
Work	ft-lb	1.356 J

Consider the following examples to illustrate the use of this conversion table.

EXAMPLE 1.6

(a) Determine the SI equivalent of a force of 112 kip.

Solution:

Using the data listed in Table 1.4, we write:

$$F = (112 \text{ kip})\ (4.448 \text{ kN/kip}) = 498.2 \text{ kN}$$

(b) Determine the SI equivalent of the mass of 5.3 slugs.

Solution:

Using the data listed in Table 1.4, we write:

$$m = (5.3 \text{ slug})\ (14.59 \text{ kg/slug}) = 77.33 \text{ kg}$$

EXAMPLE 1.7

(a) Determine the U. S. Customary equivalent of a force of 284 N.

Solution:

Using the data listed in Table 1.4, we write:

F = (284 N) (lb/4.448 N) = 63.85 lb

(b) Determine the U. S. Customary equivalent of a mass of 36.7 kg.

Solution:

Using the data listed in Table 1.4, we write:

m = (36.7 kg) (slug/14.59 kg) = 2.515 slug

In solving these exercises, we demonstrated the technique employing data from Table 1.4.

A second technique, for converting units from one system to another, is based on multiplying the term to be converted by a combination of units, which equals one. Let's consider a simple conversion from 16 MPa to its equivalent value expressed in terms of psi.

$$\begin{array}{cccccc} & 1^{st} & 2^{nd} & 3^{rd} & 4^{th} & 5^{th}\ \ 6^{th} \\ 16\ \text{MPa} \times & \dfrac{10^6\text{Pa}}{\text{MPa}} \times & \dfrac{\text{N/m}^2}{\text{Pa}} \times & \dfrac{\text{m}^2}{(39.37)^2\text{in}^2} \times & \dfrac{\text{lb}}{4.448\ \text{N}} \times & \dfrac{\text{psi}}{\text{lb/in}^2} = 2{,}321\ \text{psi} \end{array}$$

Let's examine this relation term by term:

1. The first term in the expression is a stress of 16 MPa. Recall that MPa is the unit for stress in the SI system.
2. The next unit term converts the metric prefix M to 10^6. Examine Table 1.3 for definitions of the metric prefixes.
3. The next unit term converts Pa to N/m^2.
4. The next unit term converts m to in.—note both the conversion factor and units are squared.
5. The next unit term converts N to lb.
6. The last term before the equal sign recognizes the symbol psi for lb/in^2.

If you find that a conversion table is not convenient and you are able to remember the numbers associated with a few basic conversions, multiplication by a number of unit terms is an excellent method for converting units from one system to another.

EXAMPLE 1.8

Using the unit multiplication method convert the stress of 1.5 ksi to its equivalent in the SI system of units.

Solution:

$$1.5\ \text{ksi} \times \frac{10^3\text{psi}}{\text{ksi}} \times \frac{\text{lb/in}^2}{\text{psi}} \times \frac{4.448\ \text{N}}{\text{lb}} \times \frac{(39.37)^2\text{in}^2}{\text{m}^2} \times \frac{\text{Pa}}{\text{N/m}^2} \times \frac{\text{M}}{10^6} = 10.34\text{MPa}$$

This approach is longer than the more direct conversion method. However, if you do not remember that a stress of 1 ksi is equivalent to 6.895 MPa, it provides a means of performing the conversion using only the most fundamental of the conversion factors.

1.11 SIGNIFICANT FIGURES

With the advent of the hand held calculator, we usually obtain solutions with ten or more digits. These results are misleading, because the data used in the formulas to calculate these ten digit results are rarely accurate to more than 0.2% or one part in 500. To avoid the implication of fictitious accuracies, we recommend that the results from your calculator be written with four significant figures. This practice yields a computational accuracy of 1/1,000 or 0.1% that is consistent with the accuracy of the physical data and the analytical model upon which the formula is based.

Let's consider several numerical results and convert them to results acceptable in engineering practice.

EXAMPLE 1.9

Convert the following ten digit numerical results shown in the table below to a format with four significant figures.

Solution:

Table E1.9
Examples for writing Numbers with four significant figures

Ten Digit Number	Four Significant Figures	Ten Digit Number	Four Significant Figures
(a) 6.142857143	6.143	(f) 0.182926829	0.1829
(b) 0.000304136	3.041×10^{-4}	(g) 1226308544	1.226×10^{9}
(c) 0.026941846	0.02694	(h) 1628720.675	1.629×10^{6}
(d) 308.5225000	308.5	(i) 0.0000005984	5.984×10^{-7}
(e) 22.76923077	22.77	(j) 88.62508918	88.63

With reference to the results in Table E1.7, note the procedure for each entry:

(a) We rounded the fourth digit up because the fifth digit was larger than 5.
(b) We retained the first four numbers after the leading zeros. The zeros were accommodated by using 10^{-4}.

(c) We retained the first four numbers after the leading zero. Because only one zero was involved, we retained it in the representation.
(d) We maintained the fourth digit because the fifth digit was less than 5.
(e) We rounded the fourth digit up because the fifth digit was larger than 5.
(f) We maintained the fourth digit because the fifth digit was less than 5.
(g) We retained the first four numbers because the fifth digit was less than 5. The very large number was represented by using 10^9.
(h) We rounded the fourth digit up because the fifth digit was larger than 5. The very large number was represented by using 10^6.
(i) We retained the first four numbers after the leading zeros. The zeros were accommodated by using 10^{-7}.
(j) We rounded the fourth digit up because the fifth digit was larger than 5.

1.12 SCALARS, VECTORS AND TENSORS

In mechanics we deal with three types of quantities — **scalars, vectors** and **tensors**. In everyday life it is not essential to recognize the difference among them. If we have true friends, water, food, shelter and some money, all is well. All of these items are scalar quantities. For a scalar we need only to count the number in describing it. For example, with money we count it, say $62. Many of the quantities listed in Table 1.4, and encountered in engineering are scalar including:

- Area
- Energy
- Length
- Mass
- Power
- Volume
- Work

Scalar quantities are the easiest of the three quantities with which to work. We use simple arithmetic to add, subtract, multiply and divide in manipulating scalar quantities. No additional operations are required. This is not the case with vectors and tensors. Additional mathematical operations and descriptions must be introduced to manipulate equations containing vectors or tensors.

We must deal with both vector and tensor quantities in mechanics. They are not difficult, but we need to recognize that they are not scalar quantities. We also need mathematics higher than arithmetic to deal with them. Let's start with quantities that must be described with vectors. Vector quantities require two descriptors — magnitude and direction. The magnitude indicates the size of the quantity, and the direction gives its orientation. Quantities cited in Table 1.4, which require vector representation for their complete description include:

- Acceleration
- Force
- Impulse
- Moments
- Velocity

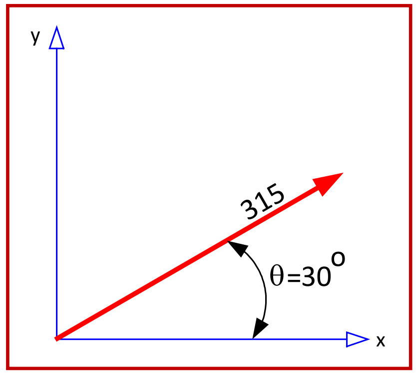
We have already discussed gravitational forces. Our weight is the magnitude of the force due to the pull of gravity on the mass of our body. The direction of a gravitational force, relative to an Earth bound coordinate system, is toward the center of the Earth.

We represent a typical vector quantity, such as a force, with an arrow as shown in Fig. 1.14. The length of the arrow is proportional to the magnitude of the force, and its inclination relative to the x-y coordinate system gives its direction. We will use vectors extensively in our studies of Statics, and learn to accommodate both their magnitude and directions in the solution of many different types of equilibrium problems.

Vectors are independent of the orientation of the x-y coordinate system. However, the scalar value of the vector's components in the x and y directions are dependent on the orientation of the reference coordinate system. This property of vectors is important, when coordinate systems are defined in solving equilibrium problems. Orientation of the axes can be made to simplify the solution to the problem without changing its physical description.

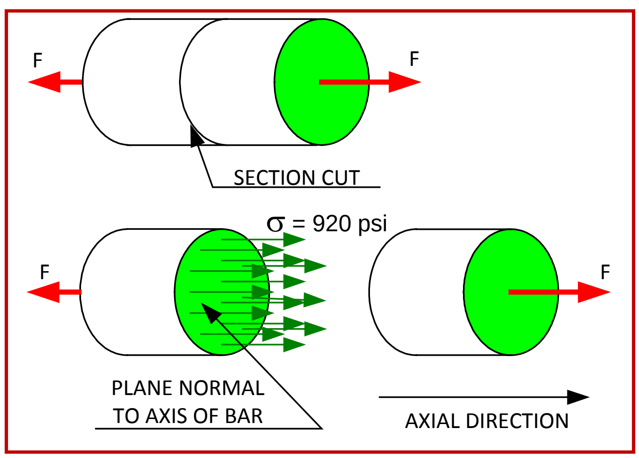
Tensor quantities are more difficult to describe than vector quantities, because their complete description requires information about three characteristics. Like vectors, we must specify the magnitude and direction. In addition we must specify the orientation of the plane upon which the tensor quantity acts. Stresses are tensor quantities. We demonstrate the three characteristics — magnitude, direction, and the orientation of the plane upon which the stresses act in Fig. 1.15.

Fig. 1.14 A force vector with a magnitude of 315 lb and direction of 30^{o} relative to the x-axis.



The round bar, shown in Fig. 1.15, is loaded with forces F in the axial direction. A section cut exposes an internal surface (plane) of the bar normal to its axis. Stresses with a magnitude of 920 psi act in the axial direction.

Fig. 1.15 Stresses of 920 psi act in the direction of the axis of the bar. The stresses are acting on the plane normal (perpendicular) to the bar's axis.



1.13 SUMMARY

In Statics we assume that the body is perfectly rigid — it does not deform under load. With this assumption, a free body diagram and the equations of equilibrium, we are able to solve different types of problems arising in Statics. Mechanics of Materials is an extension of Statics, where very small changes in the geometry of the body due to deformations are used to determine stresses and strains.

A very brief review of the history of mechanics is given to show that both Statics and Mechanics of Materials have been well understood for many years. The contributions of Archimedes, Da Vinci, Hooke, Newton, and Jacob Bernoulli are mentioned.

Newton's three laws of motion that form the foundation for mechanics have been described. Newton's laws lead to Eqs. (1.2) and (1.3) which are so important that they are repeated here:

$$\boxed{\Sigma \mathbf{F} = 0 \qquad (1.2)}$$

$$\boxed{\Sigma \mathbf{F} = m\mathbf{a} \qquad (1.3)}$$

Forces are described in considerable detail. Because forces due to gravity and pressure are so common, we provided the equations used to determine their magnitudes. The force or weight due to gravity is:

$$\boxed{F = m_b\, g = W \qquad (1.8)}$$

The force due to pressure is:

$$\boxed{F = p\, A \qquad (1.9)}$$

Both external and internal forces have been discussed. External forces are due to applied loads and reactions from supports. Internal forces are due to stress distributions found on planes within the body. We visualize these stresses and internal forces on planes exposed with imaginary section cuts.

External moments have been defined together with a method for determining the sense (sign) of the moment. We also discussed internal moments and showed that they were due to stresses developed on planes within the body. We visualize these stresses, internal forces and moments on planes exposed with imaginary section cuts.

The basic quantities and their units that are employed in Statics and Mechanics of Materials have been given. A table listing the conversion factors for the SI and U. S. Customary systems has been included. Examples showing different techniques for converting units from one system to the other are presented.

REFERENCES

1 Timoshenko, S. P., History of Strength of Materials, Dover edition, Dover Publications, New York, NY 1983.
2 Todhunter, I. and K. Pearson, History of the Theory of Elasticity and Strength of Materials, Cambridge University Press, Cambridge, U. K., 1893.
3 Uccelli, A., Leonardo da Vinci, Reynal and Co., New York, NY 1956.
4 Newton, Sir Isaac, Philosophiae Naturalis Principia Mathematica, 1687.

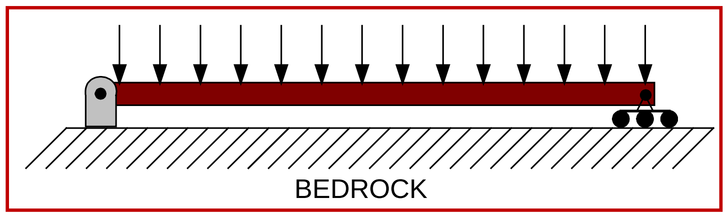
CHAPTER 2

EQUILIBRIUM AND MODELING

2.1 INTRODUCTION

Equilibrium is an extremely important concept, because it provides us with an approach for determining the unknown forces that act on and within a body. We understand that a body is in equilibrium if it is at rest or moving with a constant velocity. For example, the beam shown in Fig. 2.1 is at rest, because the loads acting downward are resisted by the reactive forces produced by the supports located at each end. The entire assembly is fixed to bedrock.

Fig. 2.1 A simply supported beam is at rest and in equilibrium.



Bodies that move may or may not be in equilibrium. If a body is in equilibrium and in motion, it moves with a constant velocity and a constant **momentum**[1]. The effect of unbalanced forces ($\sum F \neq 0$) acting on a body is to change its momentum by altering its velocity. The rate of that change is proportional to the unbalanced forces. Therefore, when a body moves at constant velocity (zero acceleration) its momentum is constant and the body is in equilibrium.

The automobile in Fig. 2.2 is travelling at a constant velocity and it is in equilibrium, because the forward thrust due to the action of the wheels on the pavement is exactly equal to the rolling friction and aerodynamic drag forces that tend to impede the forward motion of the auto. The sum of the forces in the direction of motion is zero, the momentum is constant, the acceleration is zero, and the auto is in equilibrium although it may be moving at a velocity of 70 MPH.

Fig. 2.2 An automobile traveling at constant velocity is in equilibrium.

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[1] Momentum, $\mathcal{M}$ a vector quantity, is defined as $\mathcal{M} = m\mathbf{v}$; where **v** the velocity of the body gives the direction of the momentum vector..

2.2 EQUATIONS OF EQUILIBRIUM

In Chapter 1, two equations of equilibrium were described:

$$\Sigma \mathbf{F} = 0 \qquad \text{and} \qquad \Sigma \mathbf{M_O} = 0$$

These relations are represented in a vector format; however, they may also be represented in scalar format by using Cartesian components of the forces and the moments, as shown below:

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0; \qquad \Sigma F_z = 0 \qquad (2.1)$$

$$\Sigma M_x = 0; \qquad \Sigma M_y = 0; \qquad \Sigma M_z = 0 \qquad (2.2)$$

The two vector equations of equilibrium are equivalent to the six scalar equations of equilibrium. We will develop many of the solutions to problems arising in both Statics and Mechanics of Materials by using Eqs. (2.1) and (2.2). The decision regarding which form of the equilibrium equations to employ will be left to your discretion, although in many cases you will find the scalar equations easier to apply.

Although there are six scalar equations of equilibrium, it is not always necessary to use all of them to solve an equilibrium problem. By classifying different force systems that act on a body, we can identify only those equations that provide relevant information for solving equilibrium problems. This classification greatly simplifies the associated equilibrium problems. Force systems are classified as follows:

- Non-coplanar and non-concurrent.
- Non-coplanar and concurrent.
- Coplanar and non-concurrent.
- Coplanar and concurrent.

Let's now examine the relevant equations of equilibrium for each of these force system.

2.2.1 Non-coplanar, Non-concurrent Force Systems

The three-dimensional force system, illustrated in Fig. 2.3, is **non-coplanar**, because forces with components oriented in the x, y, and z directions act on the body. It is **non-concurrent**, because the lines of action of the forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$ do not intersect at a common point. Forces F_1, F_2 and F_3 produce moments, and clearly there are components of the forces in all three directions. When the body in Fig. 2.3 is in equilibrium, the directions and magnitudes of $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$ must satisfy the **six** Cartesian component equations of equilibrium given in Eq. (2.1) and Eq. (2.2).

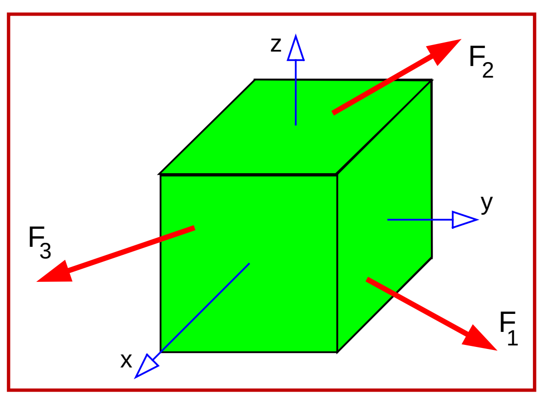
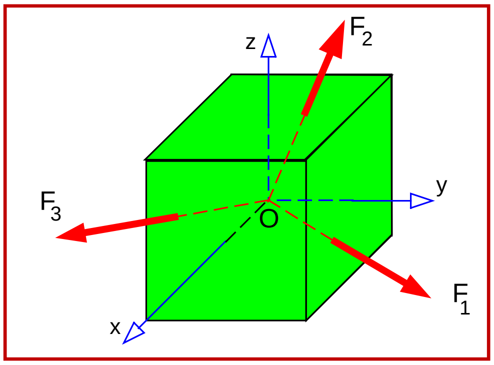


Fig. 2.3 An illustration of a non-coplanar, non-concurrent force system.

2.2.2 Non-coplanar, Concurrent Force Systems

The three-dimensional force system, illustrated in Fig. 2.4, is **non-coplanar**, because forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$ acting on the body exhibit components in the x, y and z directions.

Fig 2.4 Example of a **non-coplanar**, **concurrent** force system.



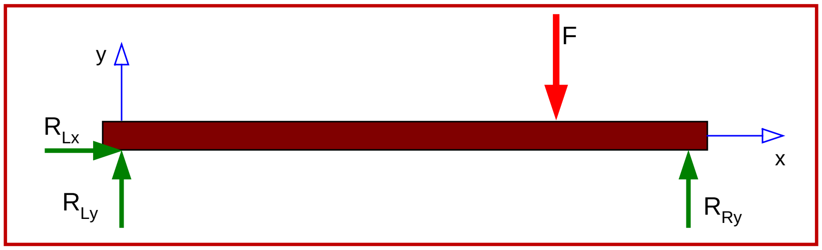
This force system is **concurrent,** because all of the forces pass through a common point (the origin of the Cartesian coordinate system in this illustration). The force system does not produce moments about the origin, because all the forces pass though the origin and the equilibrium relation $\Sigma\mathbf{M}_O = 0$ is independent of the magnitude and direction of the forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$. However, these forces must satisfy the three remaining Cartesian equations of equilibrium:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \qquad (2.1)$$

2.2.3 Coplanar, Non-concurrent Force Systems

A beam loaded, as shown in Fig. 2.5, is an example of a coplanar non-concurrent force system. Note the supports have been replaced by reaction forces and an x-y coordinate system has been defined.

Fig. 2.5 A coplanar, non-concurrent force system.



Clearly, all of the forces acting on the beam lie in the x-y plane, making this system **coplanar**. Three of the forces are parallel and their lines of action cannot intersect; hence, the system is **non-concurrent**. In this case, the forces acting on the body, F, R_{Lx}, R_{Ly} and R_{Ry}, must satisfy three of the six equations of equilibrium, namely:

$$\Sigma F_x = R_{Lx} = 0, \qquad \Sigma F_y = 0, \qquad \text{and} \qquad \Sigma M_z = 0 \qquad (a)$$

The remaining three equations of equilibrium are satisfied automatically.

2.2.4 Coplanar, Concurrent Force Systems

A **coplanar, concurrent force system** is shown in Fig. 2.6.

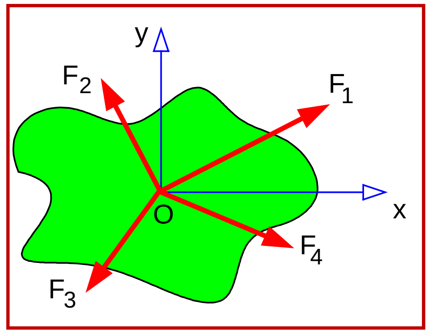


Fig. 2.6 A coplanar and concurrent system of forces.

This force system is **coplanar**, because all of the forces acting on the body lie in the x-y plane. It is **concurrent**, because the lines of action of all the forces pass through point O. This is the simplest of the four force systems because equilibrium is satisfied when:

$$\Sigma F_x = 0 \qquad \text{and} \qquad \Sigma F_y = 0 \qquad \text{(a)}$$

The other four of the six equilibrium relations are satisfied automatically and provide no useful information. Due to concurrency, we recognize that no moments occur about point O and the equations $\Sigma M_x = \Sigma M_y = \Sigma M_y = 0$ are satisfied regardless of the values assigned to the forces. Also, because the forces are coplanar, they all lie in the x-y plane; hence, $\Sigma F_z = 0$.

2.3 MODELING

The equations of equilibrium that were described in the previous section apply to:

- A single body or member.
- A structure made of several members.
- A portion of a multi-member structure formed by a section cut.
- A part of a body or structure that is been formed by two or more section cuts.
- Any size element removed from a body by one or more section cuts.

Before utilizing the equations of equilibrium, we must first construct a model of the structure being analyzed. Modeling a structure involves the construction of a **Free Body Diagram (FBD),** which is a geometric representation of the member being analyzed. The purpose of the model is to simplify the physical representation of the structure by omitting fine details that are not necessary for solving an equilibrium problem.

An example for the construction of a FBD is illustrated in Fig. 2.7. The beam is subjected to a uniformly distributed load of magnitude q (N/m), and supported near each end with simple supports. In Fig. 2.7a, we represent the uniformly distributed load with a shaded rectangle placed over the span of the beam. The uniformly distributed load may also be represented with a series of arrows, as illustrated in Fig. 2.7b. We use both of these techniques to model uniformly distributed loads. The simple supports are modeled with a pin-and-clevis on its left side and a roller arrangement on its right side. The pin-and-clevis holds the beam in a fixed position at the beam's left end, while the roller at the beam's right end permits it to expand and/or contract with changes in temperature.

A four-step procedure is employed in drawing the free body of the beam.

1. Isolate the body (a beam in this case) by removing the supports and the uniformly distributed load. We show the isolated beam in Fig. 2.7c.
2. The supports are replaced with the **reaction** loads R_{Ly}, R_{Ry} and R_{Lx}. **The pin-and-clevis at the beam's left support produces reactions loads R_{Ly} and R_{Lx} in both the x and y directions**, as shown in Fig. 2.7d. However, **the roller, which is free to rotate, produces only one reactive force R_{Ry} normal to the surface of the beam** (the y direction). At this stage of the analysis, the magnitudes of the reactive forces are not known.
3. **The uniformly distributed load q applied over the length L of the beam is replaced with a concentrated force F.** The magnitude of the force is F = (qL). This force is due to gravity and acts downward, as shown in Fig. 2.7e.
4. **Finally, we dimension the FBD and establish a coordinate system** that will facilitate the equilibrium analysis. In dimensioning the FBD, we place the concentrated force at the location of the centroid (center) of the shaded rectangular area representing the uniformly distributed load (e.g. at L/2 from the beam's left support).

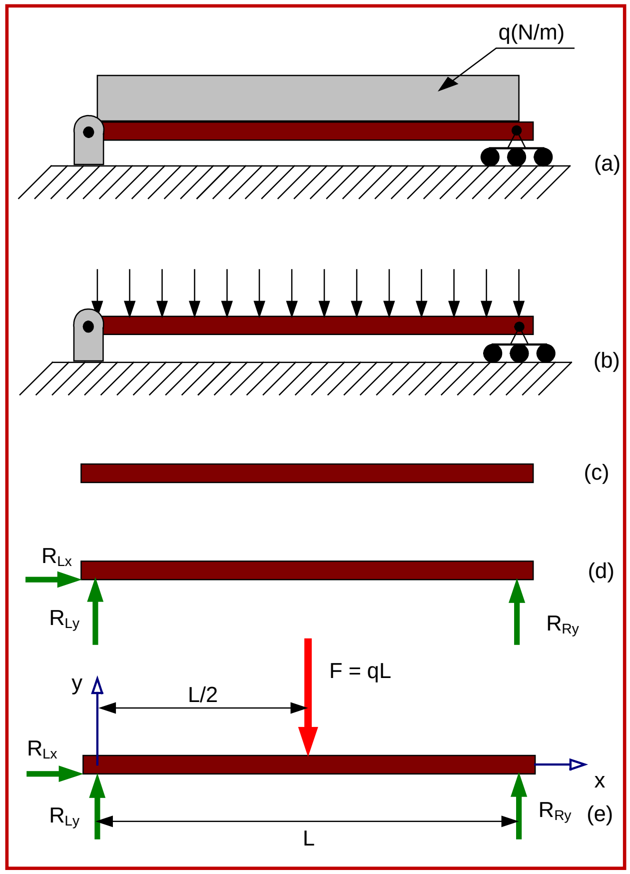


Fig. 2.7 Construction of a FBD for a simply supported uniformly loaded beam.

2.3.1 Modeling Loads

The **FBD** is a **model** of a structure or some part of a structure. To prepare the FBD, we model both the active loads that act on the structure and the reactive loads provided by its supports. Let's first consider modeling a load due to gravity, which is the most commonly encountered force. Suppose we have a block with a mass m, as shown in Fig. 2.8a. We modeled the load due to the mass of the block with a concentrated force F = mg. This concentrated force is applied at the center of the block (at its centroid). Because the force is due to gravity, the direction of the force is downward (in the negative y direction).

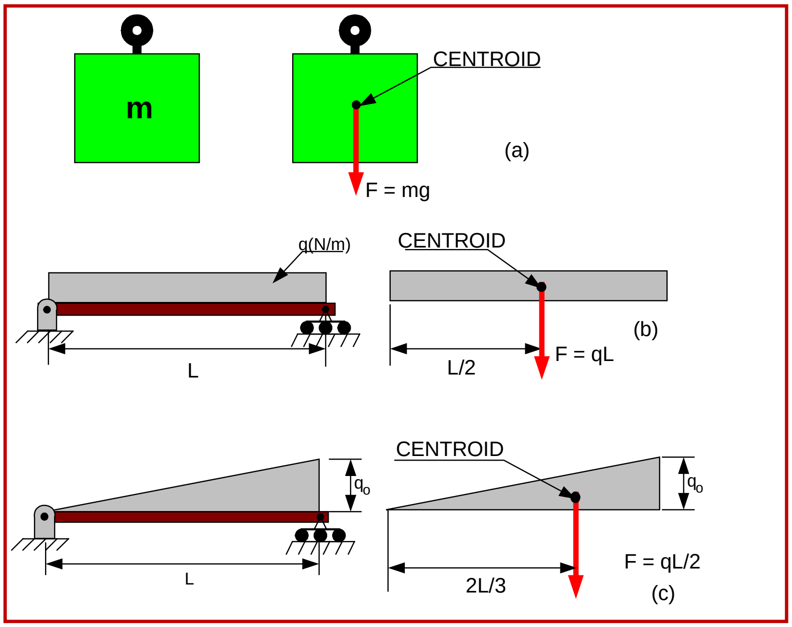


Fig. 2.8 Modeling loads due to gravity.

In Fig. 2.8 b, we encountered loads distributed over the span of a beam. In this illustration, the load is uniformly distributed with a magnitude q expressed in terms of force per unit length (N/m). We represented this uniformly distributed load with a rectangular area of height q and length L. The concentrated force F, the static equivalent of the uniformly distributed load, is given by the area of the rectangle (F = qL). The concentrated force is applied at the centroid of the rectangular area (x = L/2); because the load is due to gravity, the force acts downward.

In Fig. 2.8 c, the load is again distributed over the length of the beam, but it increases as a linear function of x (e.g. $q(x) = q_o x/L$). We represented this load with a triangular area, with an altitude q_o and a base of L. A concentrated force, the static equivalent of the distributed load, is given by the area of the triangle as $F = q_o L/2$. It is applied at the centroid of the triangle (x = 2L/3) and it acts downward.

When modeling to solve for reaction forces at structural supports, we replace a distributed load with a statically equivalent concentrated force. However, later in this chapter when we are concerned with internal moments and shear forces, it is not possible to model a distributed load with a concentrated force applied at its centroid.

Note that the drawings in Fig. 2.8 do not represent FBDs, because they are not complete. Only the modeling of the applied loads has been demonstrated. The reactive loads are absent, as well as the coordinate system.

2.3.2 <u>Modeling Supports</u>

The pin-and-clevis and roller arrangement are often used to support beams. The pin-and-clevis serves as an anchor and does not permit the beam to move in the x, y or z directions. On the other hand, the roller arrangement permits the beam to expand or contract with changes in temperature. When the pin-and-clevis and the roller are removed in the construction of a FBD, these components are replaced with the reaction forces, shown in Fig. 2.9. For the pin-and-clevis, which restrains motion in the x and y directions, reaction forces R_{Ly} (perpendicular to the bottom surface of the beam) and R_{Lx} (parallel to the bottom surface of the beam) are required to represent this support. For the roller (on a frictionless surface), a single reaction force R_{Ry} perpendicular to the bottom surface of the beam is sufficient for the representation on a FBD. Moments cannot develop with either of these supports, because the beam is free to rotate about the pin and roller both of which are considered frictionless.

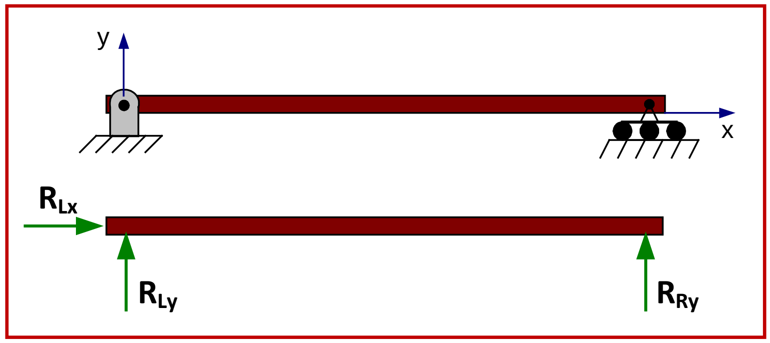


Fig. 2.9 Modeling the pin-and-clevis and the pinned roller supports.

There are many support conditions and connections to structures. In constructing FBDs, we model the structure by removing these supports and connections replacing them with one or more reactive forces and/or reactive moments. We will list several different types of supports or connections and their reactive forces and moments in the illustrations presented in Fig. 2.10.

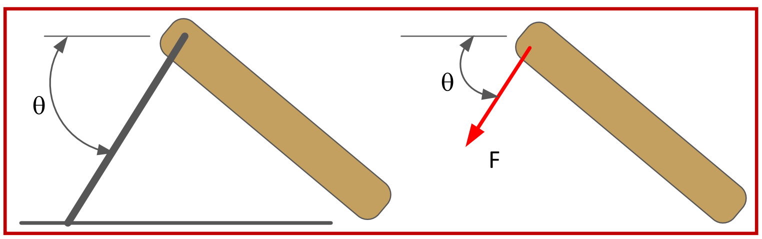


Fig. 2.10a A cable connection is represented with a single tension force (reactive) acting along the cable in the direction away from the structural element.

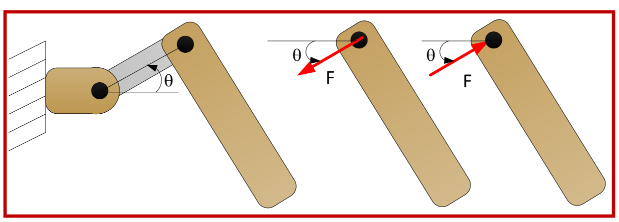


Fig. 2.10b A weightless link connection between a pin and clevis and a structural member. The link is replaced with a force along the axis of the link.

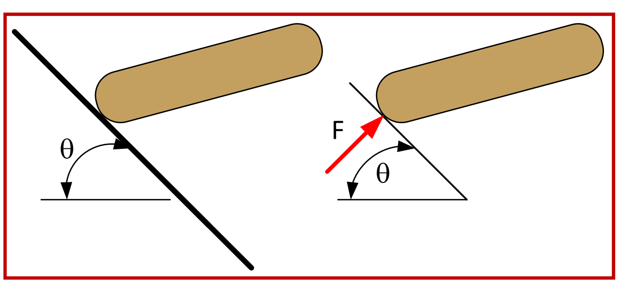


Fig. 2.10c A structural element contacting a smooth (frictionless) surface is modeled with a concentrated reactive force that acts perpendicular to the surface at the point of contact.

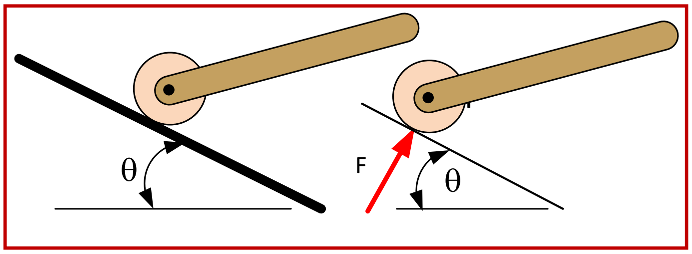


Fig. 2.10d A roller or rocker on a flat surface is represented with a concentrated reactive force that acts perpendicular to the surface at the point of contact.

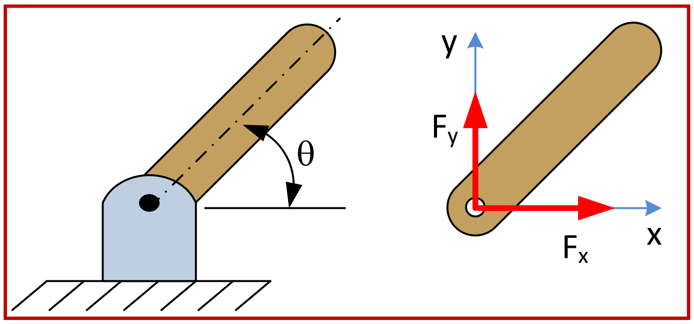


Fig. 2.10e A pin-and-clevis connection is represented with two Cartesian reactive forces perpendicular to the axis of the pin.

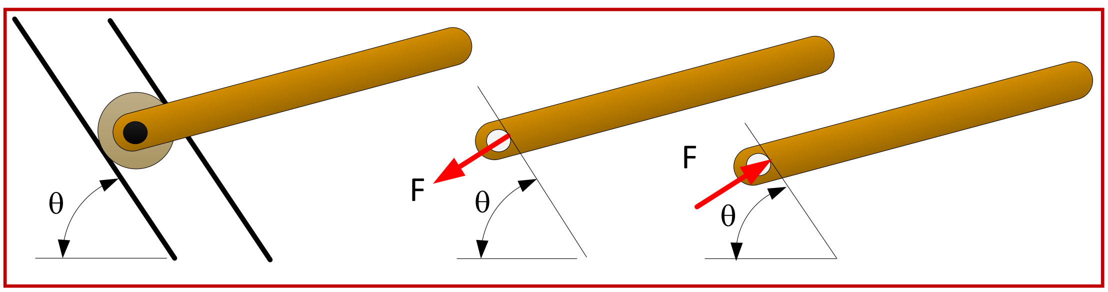


Fig. 2.10f Roller or pin in a confined smooth slot is represented with a reaction force perpendicular to the slot.

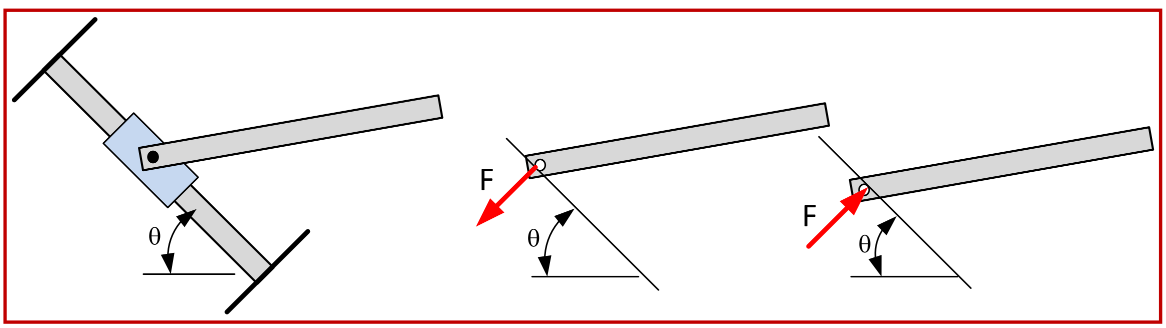


Fig. 2.10g Structural member is pin connected to a sleeve bearing on a smooth rod. The reaction force is perpendicular to the rod on which the bearing slides.

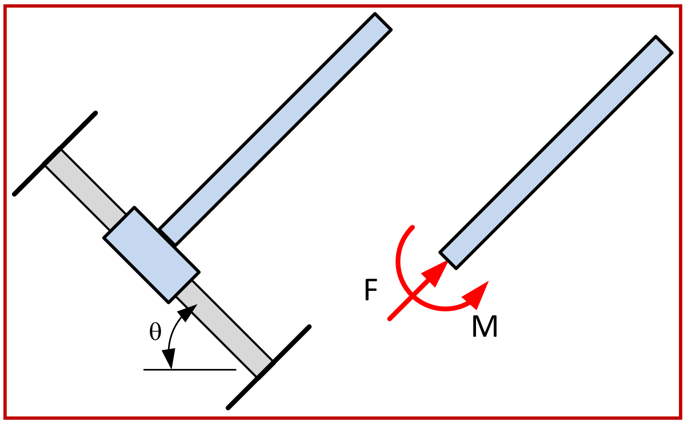


Fig. 2.10h Structural member is rigidly connected to a sleeve bearing on a smooth rod. The reaction force is perpendicular to the rod on which the bearing slides. In addition a reaction moment develops.

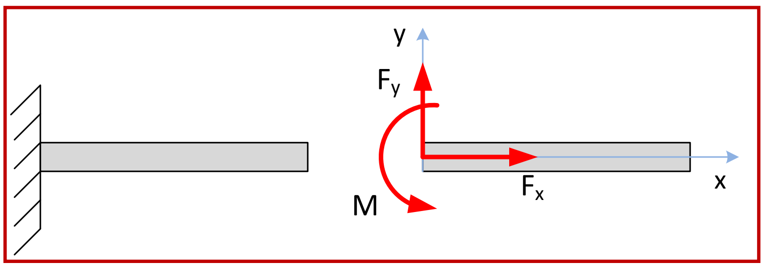


Fig. 2.10i A fixed support is represented with three possible reactions (two Cartesian reactive forces and one Cartesian reactive moment), when the applied forces are in the x-y plane.

EXAMPLE 2.1

Model the structure, shown in Fig. E2.1, by drawing a FBD of the ring that is used to attach the cables to the weight.

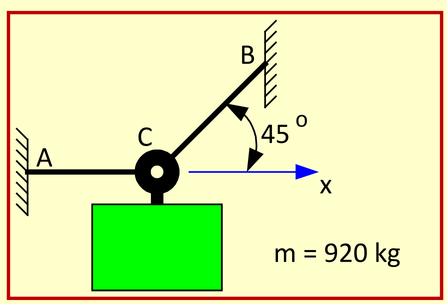


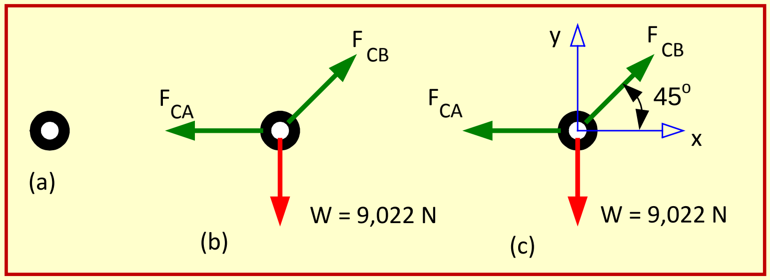
Fig. E2.1

Solution:

The reactive tension forces developed in the cables CA and CB maintain the 920-kg mass in equilibrium. We will determine the tension forces in the cables later, but the first step in solving for these forces is to construct a complete FBD. In this example, the small ring located at point C is the structural element used in constructing the free body of interest.

Step 1: Isolate the body. The isolated ring is shown in Fig. E2.1a.

Fig. E2.1a-c



Step 2: Replace the cables with force vectors, as shown in Fig. E2.1b. The forces F_{CA} and F_{CB} are of unknown magnitude; however, they are oriented in the direction of cables CA and CB respectively.

Step 3: Replace the mass of the block with a force vector acting vertically downward through the center of gravity of the block. The weight of the mass is given by:

$$W = (920 \text{ kg})(9.807) \text{ N/kg} = 9{,}022 \text{ N}.$$

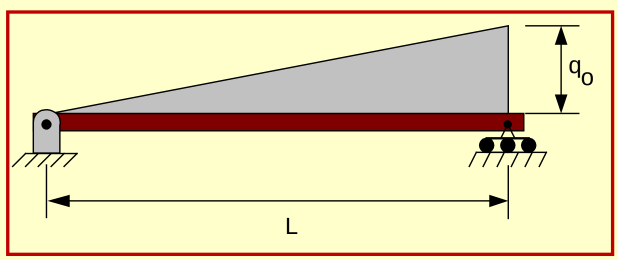
Step 4: Dimension the FBD and establish a coordinate system. Because the force system is concurrent, linear dimensions are not relevant. We show the 45° angle that F_{CB} makes with the x-axis. An x-y coordinate system is placed on the FBD, with its origin located at the center of the ring.

Let's consider another example showing the procedure for drawing FBDs. In this instance, we will draw a FBD for a coplanar, non-concurrent force system.

EXAMPLE 2.2

The beam, shown in Fig. E2.2, is subjected to a distributed load that increases from zero at its left-hand support to q_o at its right-hand support. Draw the FBD and show all the external forces acting on the beam.

Fig. E2.2



Solution:

Again we model the distributed loading with arrows, as shown in Fig. E2.2a. Then we utilize the four-step procedure to prepare the FBD.

Step 1: Isolate the body. The isolated beam is shown below in Fig. E2.2b.

Step 2: The supports are replaced with the reaction loads R_{Ly}, R_{Ry} and R_{Lx}. At this stage of the analysis, the magnitude of these forces is not known. However, at the roller support, the direction of the reaction forces R_{Ry} is normal to the surface of the beam, as shown in Fig. E2.2c. At the pin-and-clevis, reaction forces R_{Lx} and R_{Ly} are shown in the x and y directions, respectively.

Step 3: Replace the linearly distributed load with a concentrated force F, as shown in Fig. E2.2d. The area under the triangle representing the linearly increasing distributed load gives the magnitude of F. Hence;

$$F = (1/2)\, q_o\, L \qquad \text{(a)}$$

The force F is positioned at x = (2/3) L, which corresponds to the location of the centroid of the area of a triangle. The force is directed downward to coincide with the direction of the gravitational field.

Step 4: Dimension the FBD and add the coordinate system shown in Fig. E2.2d.

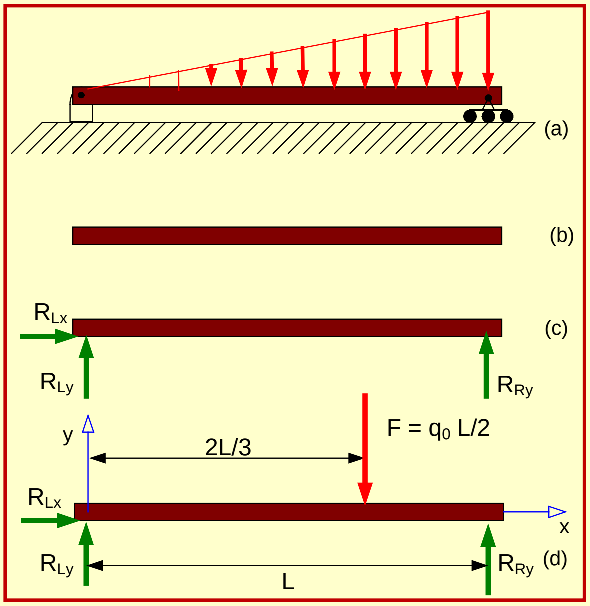


Fig. E2.2 a-d

The FBD is complete because:

1. All the external forces acting on the beam are shown.
2. The forces are located in the correct positions.
3. All the required dimensions are given.
4. A coordinate system is provided for reference.

You should use this checklist to determine if the free bodies that you prepare in executing the assigned problems are complete.

EXAMPLE 2.3

Three smooth (frictionless) cylinders each with a diameter D and a mass of 3.2 slugs are constrained by a U shaped channel with a width W, as illustrated in Fig. E2.4. Draw individual FBDs for cylinders A and C if the ratio W/D = 2.3.

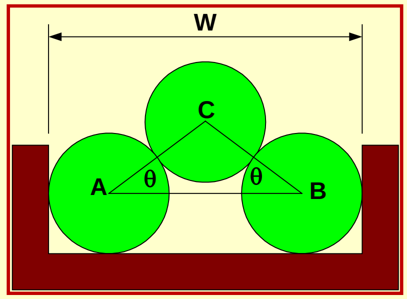


Fig. E2.3

Solution:

Before we construct the FBDs, it is necessary to determine the stacking angle θ of the cylinders. From the geometry of the cylinders in the channel, we write:

$$W = D + 2D\cos\theta \quad \Rightarrow\Rightarrow \quad \theta = \cos^{-1}\{(W)/(2D) - ½\} \qquad \text{(a)}$$

$$\theta = \cos^{-1}(1.15 - ½) = 49.46° \qquad \text{(b)}$$

Next let's determine the weight W of one of the cylinders:

$$F = mg = (3.2\ \text{slug})(32.17\ \text{lb/slug}) = 102.94\ \text{lb} \qquad \text{(c)}$$

An examination of cylinder A indicates that four forces must be placed on its FBD — three contact reactive forces and one gravitational force. The FBD for cylinder A is shown in Fig. E2.3a.

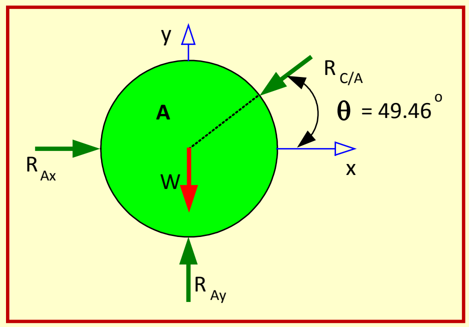


Fig. E2.3a

The reaction forces R_{Ax} and R_{Ay} are due to the surfaces of the channel contacting cylinder A. Because the cylinder is smooth (frictionless), the reaction forces are normal to the contact surface. We have used a double subscript to identify them — the first subscript identifies the cylinder in question and the second gives the direction of the force. Another reaction force $R_{C/A}$ is due to the action of cylinder C on cylinder A. It is normal to the surface of the cylinder at the point of contact. The subscript C/A indicates that cylinder C is acting on cylinder A. The angle θ = 49.46° is specified on the FBD. Finally, the weight W is shown acting downward (in the negative y direction) from the center of the cylinder.

Following similar procedures, we draw the FBD for cylinder C in Fig. E2.3b.

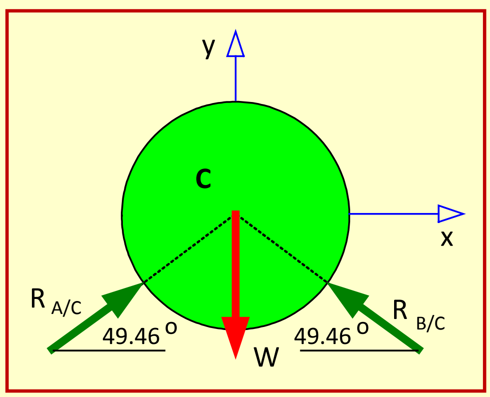


Fig. E2.3b

In this FBD, we have the two reaction forces $R_{A/C}$ and $R_{B/C}$ that are both normal to the cylindrical surfaces at the contact points. The weight W acts downward from the center of gravity of the cylinder. Let's use the checklist to ascertain if the FBDs are complete.

- Are all the forces acting on each cylinder shown? Yes!
- Are the forces located in the correct positions? Yes, we have positioned the reaction forces at the points of contact and the gravitational forces at the center of gravity of each cylinder.
- Are the required dimensions given? In this case the forces are concurrent and dimensions are not necessary. However, the directions of the reaction forces, which are required, are given by the angle θ.
- Is a coordinate system provided for reference? Yes, we have placed coordinates on both FBDs with their origins at the center of each cylinder.

EXAMPLE 2.4

A hex nut is being tightened with the wrench shown in Fig. E2.4. The dimension of the six flats on the hex nut is 40 mm. Prepare FBD diagrams for the nut and the wrench.

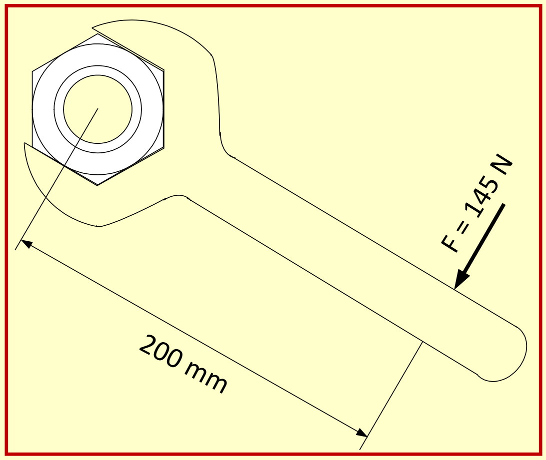


Fig. E2.4

Solution:

The free body diagrams of the nut and the wrench are shown in Fig. E2.4a and E2.4b. When the force is applied to the wrench, it contacts the hex nut at two corners producing reaction forces R_1 and R_2, which are perpendicular to the surfaces of the flats on the hex nut. The threads on the nut react with those on the bolt to produce a resistive moment (torque) M_z and a reactive force R_3. The angle defining the direction of the reaction forces acting on the flats is 60° because of the geometry of the hex nut, as shown in Fig. E2.4a. The distance between the forces is the dimension of the flats on the hex nut.

The FBD of the wrench is presented in Fig. E2.4b. The reaction forces R_1 and R_2 are applied at locations corresponding to the corners of the hex nut. They are opposite in direction of those shown on the FBD of the hex nut. The force applied to the wrench handle is shown. Dimensions locating the point of application of the force F and the reaction forces R_1 and $R_2$are provided. Angles are not shown in Fig. E2.4b, because the angles are implied by the geometry of the hex nut.

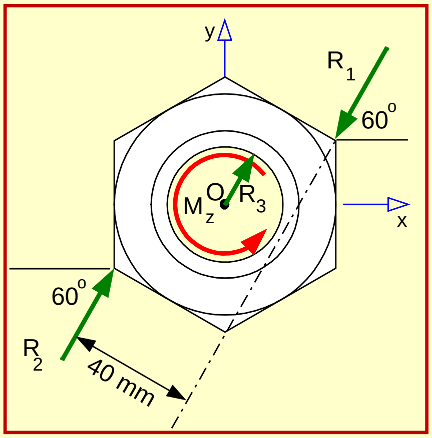


Fig. E2.4a

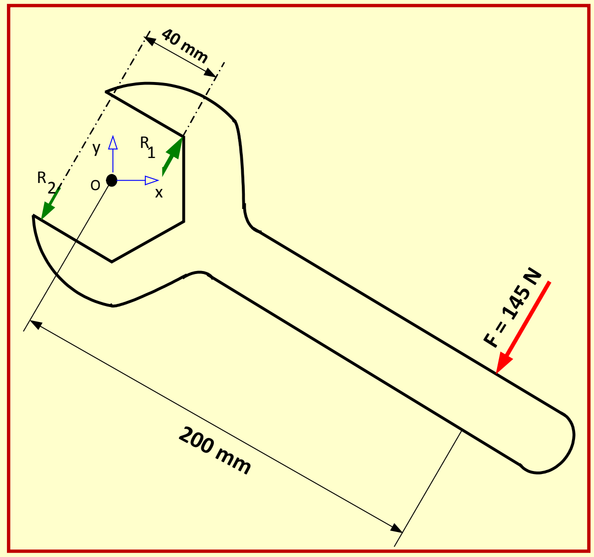


Fig. E2.4b

EXAMPLE 2.5

Prepare a FBD of a wheeled crane lifting a weight W.

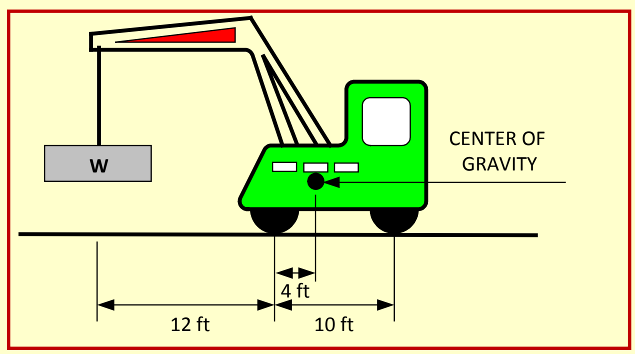
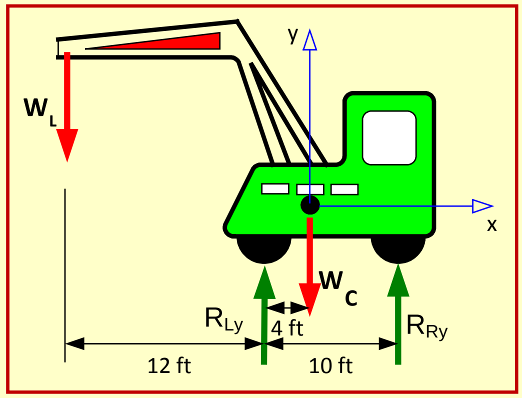


Fig. E2.5

Solution:

Let's isolate the body and apply the reaction loads R_{Ly} and R_{Ry} at the wheels, as shown in Fig. E2.5a. Next, the forces W_L and W_C due to gravity are applied at the center of gravity of the weight and the crane, respectively. The dimensions locating the points of application of the forces are specified. Finally, a Cartesian coordinate system with its origin at the center of mass of the crane is established.

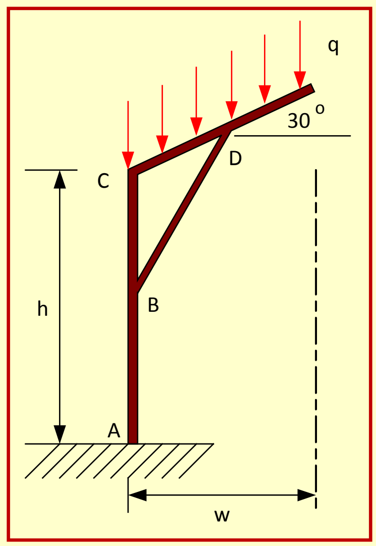
Fig. E2.5a



EXAMPLE 2.6

A small stadium for football games and other sporting events is constructed with a partial roof covering the seating arrangements. One of the many supports for the roof is illustrated in Fig. E2.6. A uniformly distributed load of q is applied to the inclined member that supports the roof. Prepare a FBD of this structural element. Assume the support is modeled as a planar two-dimensional structure.

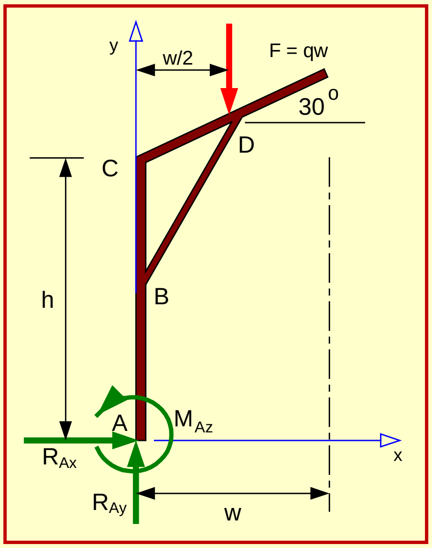
Fig. E2.6



Solution:

We first isolate the stadium component by removing the uniformly distributed load and its base support. The uniformly distributed load is replaced with a statically equivalent force F = qw applied at a distance w/2 from the vertical member. The fixed end support of the vertical member is then replaced with reaction forces R_{Ax} and R_{Ay} and a reaction moment M_{Az}. We show the moment M_{Az} as positive (counterclockwise). Because the structure is coplanar, the other reactions at point A vanish ($R_{Az} = M_{Ax} = M_{Ay} = 0$). A Cartesian coordinate system is placed on the FBD with its origin at point A. The dimension and magnitude of the applied force F is specified. The complete FBD is presented in Fig. E2.6a.

Fig. E2.6a



2.4 SOLVING FOR REACTIONS

The FBD provides a model to assist us in writing equilibrium equations that are used in the solution for unknown forces and reactions. The FBD indicates the known and unknown forces and their directions. It also provides the dimensions and angles needed to compute forces and moments. Let's consider the FBDs that we developed in a few of the previous examples, and solve for the unknown forces by using the appropriate equations of equilibrium.

EXAMPLE 2.7

Recall the 920 kg mass that was supported by cables AC and BC, as shown in the illustration in Example 2.1. In Section 2.3, we constructed a FBD for the ring that connects the weight and the cables together. For your convenience the FBD of the ring is shown again in Fig. E2.7. Solve for the unknown forces acting on the ring.

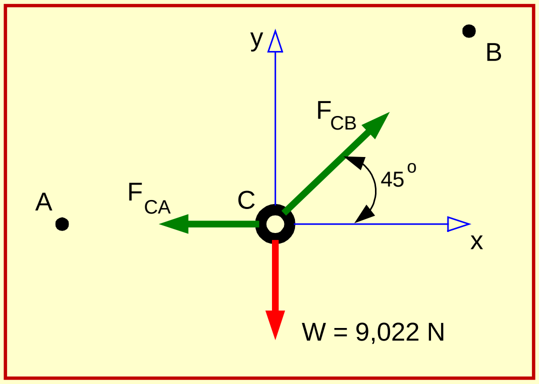


Fig. E2.7

Solution:

The solution for the unknown forces and reactions is accomplished using the equations of equilibrium with an easy six-step procedure. We demonstrate this step-by-step approach for the cable-mass arrangement represented by the FBD.

Step 1: Prepare a complete FBD of the ring as shown above.

Step 2: Classify the force system. In this example, the force system is coplanar and concurrent.

Step 3: Write the relevant equations of equilibrium. For coplanar and concurrent force systems, only two of the six equations of equilibrium provide relevant information.

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \text{(a)}$$

Step 4: Substitute the unknowns from the FBD into the two relevant equations of equilibrium shown in Eq. (a).

$$\Sigma F_y = F_{CB} \sin(45°) – 9{,}022 \text{ N} = 0 \qquad \Sigma F_x = F_{CB} \cos(45°) – F_{CA} = 0 \qquad \text{(b)}$$

Step 5: Execute the solution.

$$F_{CB} = (9{,}022 \text{ N})/(0.7071) \qquad F_{CB} = 12{,}759 \text{ N}$$

$$F_{CA} = (12{,}759 \text{ N})(0.7071) \qquad F_{CA} = 9{,}022 \text{ N} \qquad \text{(c)}$$

Step 6: Check the solution and interpret the results. In this example, the equations are simple, and we may easily determine if the numerical results are realistic by checking the signs in each of the equilibrium equations. Note that **forces directed in the negative coordinate directions are considered negative quantities** in writing the ΣF terms. You should always make certain that the forces are expressed in the appropriate units. In this problem, the mass was given in kg (SI units), and this fact dictates that the force be expressed in newtons (N). Finally, we make a sketch of the force vectors, shown in Fig. E2.7a, verifying the results.

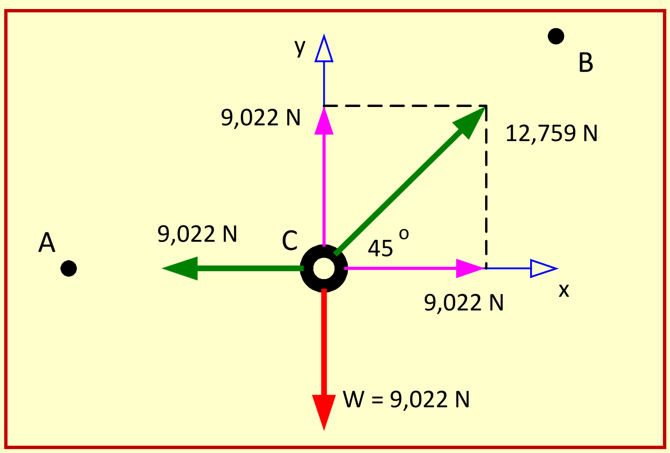


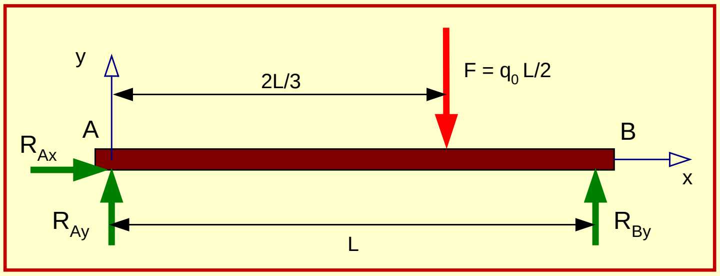
Fig. E2.7a

The 12,759 N force produces 9,022 N components in the positive x and y directions, which cancel out the two 9,022 N forces directed in the negative x and y directions. Clearly, the forces in the x and y directions are balanced, and the ring is in equilibrium.

EXAMPLE 2.8

Next consider the beam that was illustrated in Example 2.2. Recall that we constructed the FBD to model this beam as indicated below:

Fig. E2.8



Let's execute the six-step procedure to solve for the unknown magnitudes of the reaction forces R_{Ax}, R_{Ay} and R_{By}.

Solution:

Step 1: Prepare a complete FBD of the beam, as shown above. Note the force F was determined from the area of the triangle representing the linear distributed load as:

$$F = q_o\, L/2 \qquad \text{(a)}$$

Step 2: Classify the force system. In this problem, the force system is coplanar and non-concurrent.

Step 3: Write the relevant equations of equilibrium. For coplanar and non-concurrent force systems, only three of the six equations of equilibrium are relevant.

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0 \qquad \text{(b)}$$

To determine the moments to substitute into $\Sigma M = 0$, it is necessary to select some arbitrary point on the FBD, as a reference for determining the moments. We have selected point, A because it eliminates the moment due to the force R_{Ay} and simplifies the resulting equations.

Step 4: Substitute the unknowns from the FBD into the relevant equations of equilibrium.

$$\Sigma F_x = 0 \qquad \Rightarrow \qquad R_{Ax} = 0 \qquad \text{(c)}$$

$$\Sigma F_y = R_{Ay} + R_{By} - (1/2)\, q_o\, L = 0 \qquad \text{(d)}$$

$$\Sigma M_A = R_{By}\, L - (1/2)\, q_o\, L\, (2/3)L = 0 \qquad \text{(e)}$$

Step 5: Execute the solution by solving Eq. (e) for R_{By} yields:

$$R_{By} = q_o\, L/3 \qquad \text{(f)}$$

and Eq. (d) and Eq. (f) gives:

$$R_{Ay} = q_o\, L(1/2 - 1/3) = q_o\, L/6 \qquad \text{(g)}$$

Step 6: Check the solution and interpret the results. In this example, the solution involves solving two linear algebraic equations. We check the signs in each of the equilibrium equations. Note that **forces directed in the negative coordinate directions are considered negative quantities** in writing the ΣF terms. The **moments tending to rotate the beam counterclockwise are treated as positive quantities and those tending to produce clockwise motion are treated as negative quantities**. You should make certain that the forces and the moments are expressed in the appropriate units. In this problem, no units were specified because the applied loading and the beam's length were given as symbols q_o and L respectively. However, let's suppose that L = 24 ft and q_o = 800 lb/ft. With L and q_o specified we may determine numerical values and units for R_{Ay} and R_{By} as:

$$R_{By} = q_o\, L/3 = (800 \text{ lb/ft})(24 \text{ ft})/(3) = 6{,}400 \text{ lb}$$

$$R_{Ay} = q_o\, L/6 = (800 \text{ lb/ft})(24 \text{ ft})/(6) = 3{,}200 \text{ lb}$$

$$F = q_o\, L/2 = (800 \text{ lb/ft})(24 \text{ ft})/(2) = 9{,}600 \text{ lb} = R_{By} + R_{Ay}$$

The result for the force F provides a check on the solution, and the units for all of the results are specified.

Let's continue to develop your skills for solving equilibrium problems. For the next example, consider the cantilever beam that is shown in Fig. 2.11. A cantilever beam is a long slender structural member. It is built-in (supported so that its left hand end cannot deflect or rotate), and free (not supported in any manner) on its right-hand end. The cantilever beam, shown in Fig. 2.11, is loaded with a transverse concentrated force F at its free end.

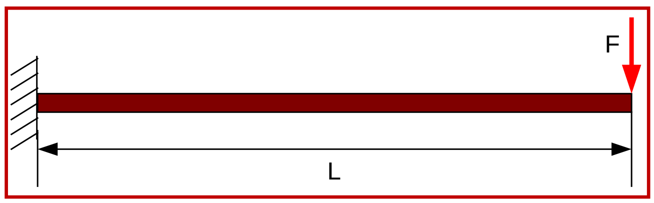


Fig. 2.11 A cantilever beam loaded at its free end with a concentrated force F.

EXAMPLE 2.9

Determine the reactions (forces and moments) at the built-in end of the cantilever beam, illustrated in Fig. 2.11.

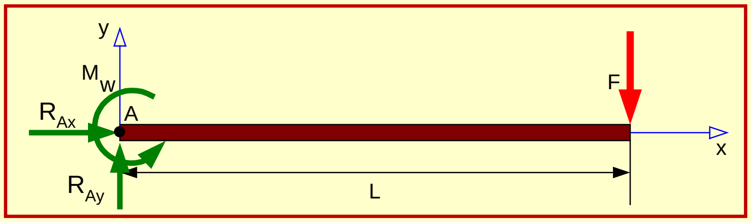
Solution:

Let's execute the six-step procedure to determine the unknown magnitudes of the reaction forces R_{Ax} and R_{Ay} and the reaction moment M_W due to the built-in support.

Step 1: Prepare a FBD to provide a model of the beam, as shown in Fig. E2.9. Note the built-in end support is removed, and replaced with the reaction forces R_{Ay} and R_{Ax} and the reaction moment M_W. The moment M_W is taken as a positive quantity (shown in the counter clockwise direction). A two-dimensional Cartesian coordinate system, with its origin at point A, was added for reference.

Step 2: Classify the force system. In this problem, the force system is coplanar and non-concurrent.

Fig. E2.9



Step 3: Write the relevant equations of equilibrium. For coplanar and non-concurrent force systems, only three of the six equations of equilibrium are relevant.

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0$$

Step 4: Substitute the unknowns from the FBD into the relevant equations of equilibrium.

$$\Sigma F_x = 0 \quad \Rightarrow \quad R_{Ax} = 0 \qquad \text{(a)}$$

With all of the applied forces in the y direction, the reaction force R_{Ax} at the built-in support is clearly zero, as shown in Eq. (a).

$$\Sigma F_y = R_{Ay} - F = 0 \qquad \text{(b)}$$

$$\Sigma M_A = M_W - F\,L = 0 \qquad \text{(c)}$$

Step 5: Execute the solution. Solving Eq. (b) for R_{Ay} and substituting in Eq. (c) yields:

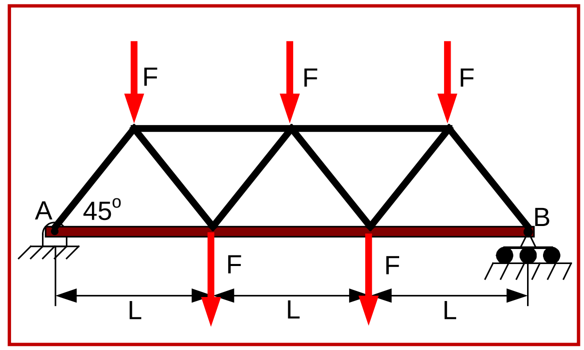
$$R_{Ay} = F \qquad M_W = FL \qquad \text{(d)}$$

Step 6: Check the solution and interpret the results. In this example, the mathematics involves solving two simple linear algebraic equations. We check the signs in each of the equilibrium equations. The forces directed in the negative coordinate directions are considered negative quantities in writing the ΣF terms. The moments tending to rotate the beam counterclockwise are treated as positive quantities, and those tending to produce clockwise motion are treated as negative quantities in writing the ΣM terms. The reaction moment M_W was assumed to be in the counterclockwise direction. The solution $M_W = FL$ is a positive quantity indicating this assumption was correct. **If we had assumed the incorrect direction, the solution of the equilibrium relation would have resulted in a negative quantity.** You should make certain that the forces and the moments are expressed in the appropriate units. In this problem, no units were specified, because the applied loading and the length of the beam were given as symbols F and L, respectively. However, let's suppose that L = 8 m and F = 32 kN. Determining numerical values and units for R_{Ay} and M_W, we find:

$$R_{Ay} = F = 32 \text{ kN} \qquad M_W = F\,L = (32 \text{ kN})(8 \text{ m}) = 256 \text{ kN-m} \qquad \text{(e)}$$

For another illustration, consider the reaction forces at the supports for the truss structure, shown in Fig. 2.12. The truss is supported at one end with a pin-and-clevis and at the other with a set of rollers. **This roller arrangement is often used to support one end of a long structure, because the roller allows movement to accommodate the changes in the structure's length due to temperature changes.** With the roller support, the truss in Fig. 2.12 is free to expand and contract with temperature and thermal stresses, due to constraint, do not develop.

Fig. 2.12 The truss is loaded at the intersection of its members with a load of F.



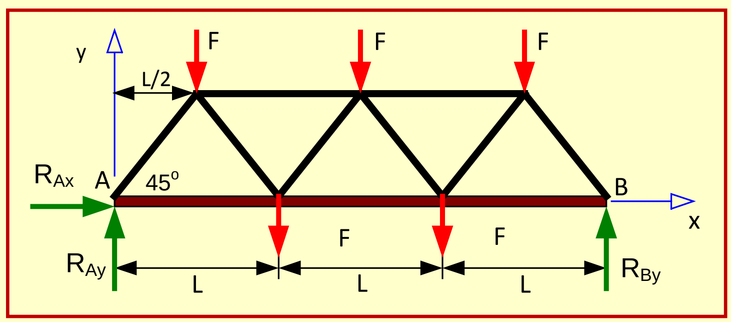
EXAMPLE 2.10

Determine the reaction forces at the supports for the truss presented in Fig. 2.12.

Solution:

Step 1: Prepare a FBD to provide a model of the truss, as shown in Fig. E2.10. The supports were removed and replaced with the reaction forces R_{Ax}, R_{Ay} and R_{By}. A two-dimensional Cartesian coordinate system was added for reference.

Fig. E2.10



Step 2: Classify the force system. In this instance, the force system is coplanar and non-concurrent.

Step 3: Write the relevant equations of equilibrium. For a coplanar and non-concurrent force system, only three of the six equations of equilibrium are relevant.

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0$$

Step 4: Substitute the unknowns from the FBD into the relevant equations of equilibrium.

$$\Sigma F_x = 0 \quad \Rightarrow \quad R_{Ax} = 0 \tag{a}$$

$$\Sigma F_y = R_{Ay} + R_{By} - 5F = 0 \tag{b}$$

$$\Sigma M_A = R_{By}\,(3L) - F(L/2) - FL - F(3L/2) - F(2L) - F(5L/2) = 0 \tag{c}$$

Step 5: Execute the solution. Solving Eq. (c) for R_{By} yields:

$$R_{By} = (2.5)\,F \tag{d}$$

and Eq. (b) gives:

$$R_{Ay} = (2.5)\,F \tag{e}$$

Step 6: Check the solution and interpret the results. Again, the mathematics involves solving two simple linear algebraic equations. We check the signs in each of the equilibrium equations. The moments tending to rotate the truss counterclockwise are treated as positive quantities, and those tending to produce clockwise motion are treated as negative quantities in writing the ΣM terms. Once again, if we had assumed the incorrect direction for the reaction forces, the solution of the equilibrium relation would have been a negative quantity. You should make certain that the forces and the moments are expressed in the appropriate units. In this problem, no units were specified because the applied loading and the lengths of the truss segments are given as symbols F and L, respectively. However, let's suppose that L = 8 m and F = 28 kN. Determining numerical values and units for R_{Ay} and R_{By}, we find:

$$R_{By} = R_{Ay} = (2.5)\,F = (2.5)(28) = 70 \text{ kN} \qquad \text{(f)}$$

Notice that the reaction forces are equal. We expected this result, because the geometry of the truss and its loading were symmetric. Also recognize that the reaction forces are independent of the length of the truss, because the length L of the truss cancelled out of Eq. (c) for the moment M_A.

EXAMPLE 2.11

Using the FBDs presented in Fig. E2.11a and Fig. E2.11b, determine the contact forces between the cylinders described in Example 2.3.

Solution:

Step 1: Prepare complete FBDs of the cylinders A and B, as shown in Fig. E2.11a and Fig. E2.11b.

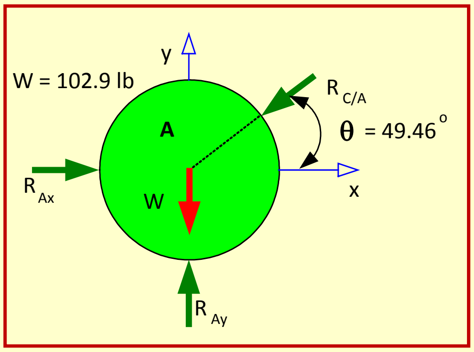


Fig. E2.11a

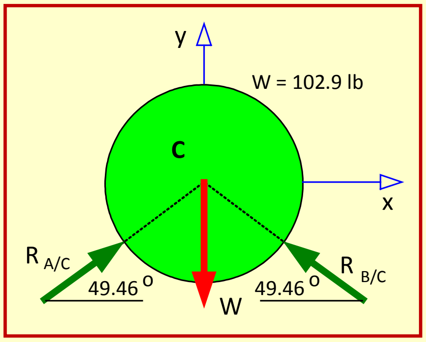


Fig. E2.11b

Step 2: Classify the force system. The force system acting on each cylinder is coplanar and concurrent.

Step 3: Write the relevant equations of equilibrium. For coplanar and concurrent force systems, only two of the six equations of equilibrium provide relevant information.

$$\Sigma F_x = 0 \qquad\qquad \Sigma F_y = 0$$

Step 4: Substitute the unknowns from the FBD for cylinder C, presented in Fig. E2.11b, into the relevant equations of equilibrium and execute the solution.

$$\Sigma F_x = R_{A/C} \cos(49.46°) - R_{B/C} \cos(49.46°) = 0$$

$$R_{A/C} = R_{B/C} \qquad \text{(a)}$$

$$\Sigma F_y = R_{A/C} \sin(49.46°) + R_{B/C} \sin(49.46°) - 102.9 = 0$$

$$R_{A/C} = R_{B/C} = (102.9)/[(2)\sin(49.46°)] = 67.70 \text{ lb} \qquad \text{(b)}$$

Step 5: Substitute the unknowns from the FBD for cylinder A, shown in Fig. E2.11a, into the relevant equilibrium equations and execute the solution.

$$\Sigma F_x = R_{Ax} - R_{C/A} \cos(49.46°) = 0$$

$$R_{Ax} = (67.70)(0.6500) = 44.00 \text{ lb} \qquad \text{(c)}$$

$$\Sigma F_y = R_{Ay} - W - R_{C/A} \sin(49.46°) = 0$$

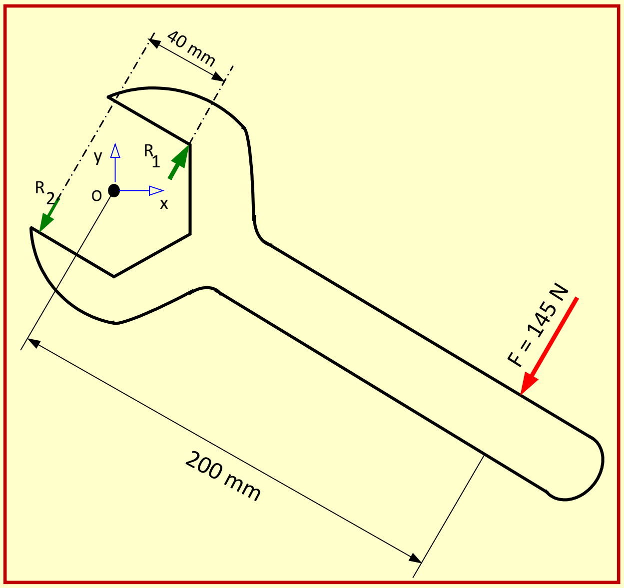
$$R_{Ay} = 102.9 + (67.70)(0.7600) = 154.3 \text{ lb} \qquad \text{(d)}$$

Step 6: Check the solution and interpret the results. In this example, the equations are not complex, and we may easily determine if the numerical results are realistic by checking the signs in each of the equilibrium equations and by repeating the calculations. Note that **forces directed in the negative coordinate directions are considered negative quantities** in writing the ΣF terms. You should always make certain that the forces are expressed in the appropriate units.

EXAMPLE 2.12

For the hex nut and wrench described in Example 2.4, determine the torque acting at the threads of the nut and the reaction forces R_1 and R_2.

Fig. E2.4b



Solution:

Let's refer to the FBD for the wrench presented in Fig. E2.4b. An examination of the FBD indicates that the force system is coplanar and non-concurrent. Writing the equilibrium relations yields:

$$\Sigma F_{\perp} = R_1 - R_2 - 145 = 0 \qquad \Sigma F_{||} = 0 \qquad \text{(a)}$$

where the subscripts $\perp$ and $||$ refer to directions perpendicular and parallel to the axis of the wrench handle, respectively.

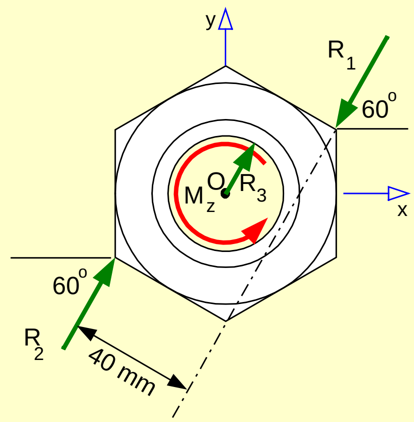
$$R_1 = R_2 + 145 \qquad \text{(b)}$$

$$\Sigma M_O = 20(R_1 + R_2) - (200)(145) = 0 \qquad \text{(c)}$$

From Eqs. (b) and (c), we solve for R_1 and R_2 to obtain:

$$R_1 = 797.5 \text{ N} \quad \Rightarrow \quad R_2 = 652.5 \text{ N} \qquad \text{(d)}$$

Fig. E2.4a



To determine the torque T applied to the nut, refer to the FBD in Fig. E2.4a.

$$\Sigma M_O = M_z - 20\,R_1 - 20\,R_2 = 0 \qquad \text{(e)}$$

$$T = M_z = (20)(797.5 + 652.5) = 29{,}000 \text{ N-mm} = 29 \text{ N-m} \qquad \text{(f)}$$

The positive sign for the moment M_z, which acts on the threads of the nut, indicates that the direction assigned to M_z in Fig. E2.4a was correct.

EXAMPLE 2.13

Determine the reactions at the wheels of the crane described in Example 2.5. The weight W_L is 1,500 lb and the wheeled crane weighs 10,000 lb.

Solution:

Let's refer to the FBD for the crane presented in Fig. E2.13. An examination of the FBD indicates that the force system is coplanar and non-concurrent.

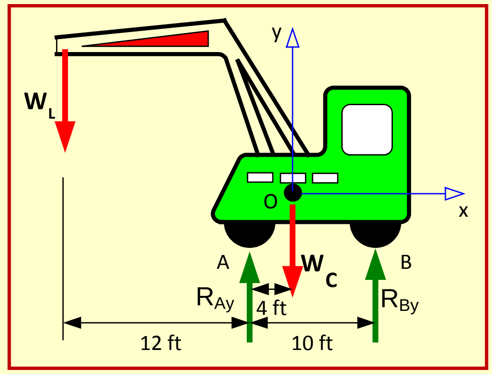


Fig. E2.13

Step 1: Writing the equilibrium relations yields $\Sigma F_x = 0$ is satisfied regardless of the values determined for the reaction forces.

$$\Sigma F_y = R_{Ay} + R_{By} - W_C - W_L = 0 \tag{a}$$

$$R_{Ay} = W_C + W_L - R_{By} \tag{b}$$

$$\Sigma M_O = (10 - 4)R_{By} - 4R_{Ay} + 16W_L = 0 \tag{c}$$

Step 2: Substituting Eq. (b) into Eq. (c) and solving for R_{Ay} and R_{By} yields:

$$R_{By} = [(2)W_C - (6)W_L]/(5) \tag{d}$$

$$R_{Ay} = [(3)W_C + (11)W_L]/(5) \tag{e}$$

Step 3: Substituting W_C = 10,000 lb and W_L = 1,500 lb into Eqs. (d) and (e) gives:

$$R_{By} = 2{,}200 \text{ lb} \quad \Rightarrow\Rightarrow \quad R_{Ay} = 9{,}300 \text{ lb} \tag{f}$$

The results obtained may be checked by substituting the numerical values for R_{Ay}, R_{By} W_C and W_L into Eq. (a) and verifying the equality.

EXAMPLE 2.14

Using the FBD in Fig. E2.14, determine the reactions at the base of the vertical member that supports the roof of the football stadium.

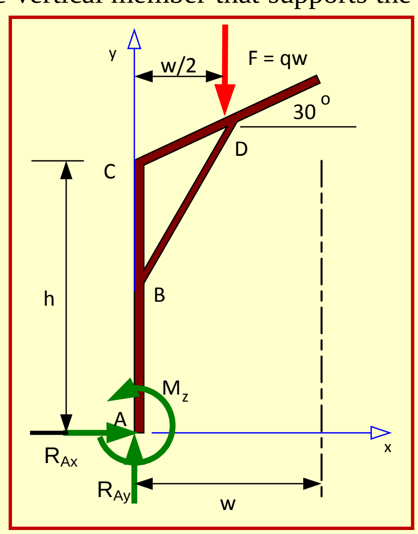


Fig. E2.14

Solution:

Step 1: The force system is classified as coplanar and non-concurrent. Accordingly we may write three equilibrium relations.

$$\Sigma F_x = 0 \quad \Rightarrow \quad R_{Ax} = 0 \qquad \text{(a)}$$

$$\Sigma F_y = R_{Ay} - qw = 0 \quad \Rightarrow \quad R_{Ay} = qw \qquad \text{(b)}$$

$$\Sigma M_A = M_z - (qw)(w/2) \quad \Rightarrow \quad M_z = qw^2/2 \qquad \text{(c)}$$

A numerical value for the uniformly distributed load q has not been provided, nor has the value for the dimension w. How are we to check the solution?

Step 2: Considering the units of the quantities in Eqs. (b) and (c) provides useful information. In Eq. (b), we know the reaction R must have a force unit F, the distributed load q a unit of F/L, and the distance w a unit of L. We then write the equation to check the units as:

$$F = (F/L)L \quad \Rightarrow \quad F = F \qquad \text{(d)}$$

Step 3: Similarly the units for the moment M in Eq. (c), must be FL. We check the homogeneity of the units in Eq. (c) by:

$$FL = (F/L)(L)^2 \quad \Rightarrow \quad FL = FL \qquad \text{(e)}$$

The results from Eqs. (d) and (e) confirm that the units in our solution are homogenous as required.

2.5 FORCES IN CABLE AND PULLEY ARRANGEMENTS

Pulleys and cables[2] are often used in lifting devices such as cranes, derricks and elevators. The pulley is like a wheel, because it rotates about a shaft with very low friction. In fact, we will assume that the pulley is frictionless. A groove is cut into its perimeter to contain the cable and prevent it from slipping off the pulley. The essential details of a pulley are illustrated in Fig. 2.13.

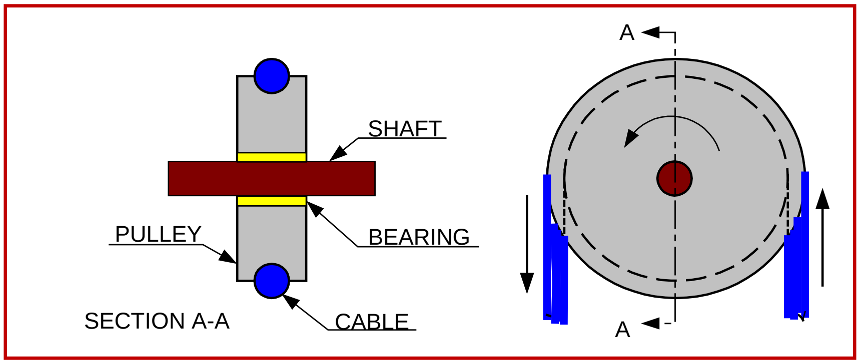


Fig. 2.13 Details of a cable and pulley assembly.

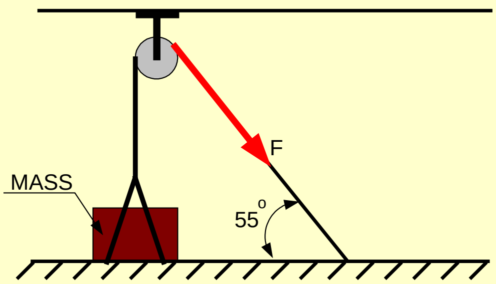
[2] A cable is like a rope except that metallic wire is used for the cable's fibers instead of hemp or some other organic material.

The pulley/cable system has two primary advantages. First, it changes the direction of a force. For instance, as we pull down on the cable at the left side of the pulley in Fig. 2.13, the cable on the right side moves up. Using this property, we may transport masses to significant heights. Second, two or more pulleys and a cable properly arranged act as a machine and amplify the force capability of a person or a mechanical device. Let's consider two different examples that illustrate the advantages of a cable/pulley system either to change the direction or to amplify a force.

EXAMPLE 2.15

The cable pulley system presented in Fig. E2.15 is being used to lift a mass of 11 slugs from the floor. Determine the force required to lift this mass and the reaction force acting on the shaft in the pulley.

Fig. E2.15



Solution:

Step 1: Prepare two FBDs of the pulley/cable system (one of the pulley and the other of the mass), as shown in Fig. E2.15a. After lifting the mass from the floor, we cut the cable from the mass and replace it with the force F_1, as shown in the FBD in Fig. E2.15a. A force W, acting downward, equal to the weight of the mass is applied. The weight of the mass is determined from Eq. (1.8) as:

$$W = mg = (11)(32.17) = 353.9 \text{ lb} \qquad \text{(a)}$$

Next remove the support attachment from the pulley and replace it with force components F_{2x} and F_{2y}. Because we have cut the cable to the mass, it is necessary to apply a force F_1 to replace the cable completing the FBD of the pulley, as shown in the FBD in Fig. E2.15a.

Step 2: Classify the force systems. For the mass, the force system is collinear (along a single line), coplanar and concurrent. For the pulley, the force system is coplanar and non-concurrent.

Step 3: Write the relevant equations of equilibrium. For the mass, only one of the six equations of equilibrium is relevant.

$$\Sigma F_y = 0$$

For the pulley, only three of the six equations of equilibrium are relevant.

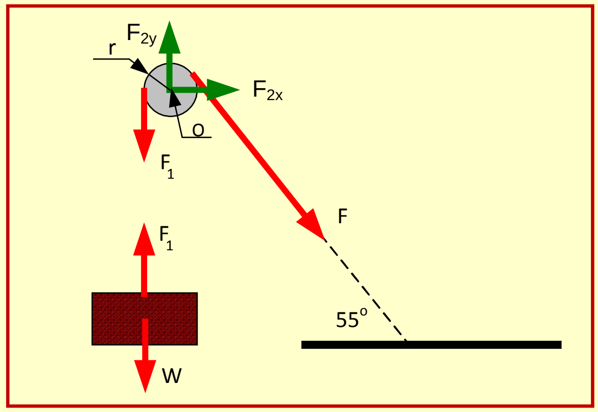
$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0$$

Steps 4 and 5: Substitute the unknowns from the FBD into the relevant equations of equilibrium for both the mass and the pulley and then execute the solutions. For the mass, we write:

$$\Sigma F_y = F_1 - W = 0$$

$$F_1 = W = 353.9 \text{ lb} \qquad \text{(b)}$$

Fig. E2.15a



The pulley, with a radius r, is assumed to be frictionless. We sum the moments about the center O of the pulley to obtain:

$$\Sigma M_O = (F_1)(r) - (F)(r) = 0$$

$$F = F_1 = 353.9 \text{ lb} \qquad \text{(c)}$$

It is clear from Eq. (c) that **the tension in the cable remains constant across the pulley and F_1 = F_2** even though the direction of the forces F and F_1 are different.

Next, consider the sum of the forces in the x and y directions for the pulley.

$$\Sigma F_x = F_{2x} + F \cos 55° = 0$$

$$F_{2x} = -(353.9) \cos 55° = -203.0 \text{ lb} \qquad \text{(d)}$$

The minus sign indicates that the direction for F_{2x} in Fig. E2.15a is not correct. The force component is actually in the negative x direction.

Next, consider the sum of the forces in the y direction.

$$\Sigma F_y = F_{2y} - F \sin 55° - F_1 = 0$$

$$F_{2y} = (353.9)(1 + 0.8192) = 643.8 \text{ lb} \qquad \text{(e)}$$

$$F_2 = [F_{2x}^2 + F_{2y}^2]^{1/2} = [(-203.0)^2 + (643.8)^2]^{1/2} = 675.0 \text{ lb}$$

Step 6. Check the solution and interpret the results. Again, the mathematics involves solving simple linear algebraic equations. We check the signs in each of the equilibrium equations and repeat the calculation. We check to ascertain that the correct units for each of the unknown forces have been assigned. We compare the forces F, F_1 and F_2 and find that their magnitudes are reasonable.

EXAMPLE 2.16

For the pulley and cable arrangement shown in Fig. E2.16, determine the force F required to maintain a mass with a weight W of 22 kN in equilibrium.

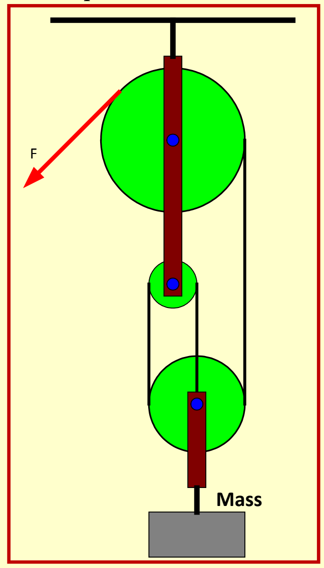


Fig. E2.16

Solution:

Step 1: Prepare a FBD of the pulley/cable system, as shown in Fig. E2.16a. Cut the three cables above the lower pulley and apply a force F to each cable. Recall that the tension in the cable is a constant across a pulley, when the friction at its shaft is zero. A force W = 400 N, acting downward replaces the mass.

Step 2: Classify the force systems. For the lower pulley, the force system is coplanar and non-concurrent.

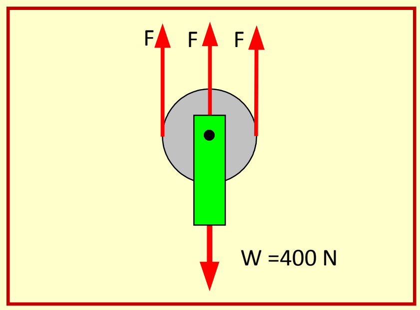


Fig. E2.16a

Step 3: Write the relevant equations of equilibrium. For the pulley, only one of the six equations of equilibrium is relevant, because no forces are applied in the x and z directions and moments are zero because the pulley pin is frictionless.

$$\Sigma F_y = 0$$

Steps 4 and 5: Substitute the unknowns from the FBD into the relevant equations of equilibrium for the pulley and then execute the solution.

$$\Sigma F_y = 3\,F - W = 0$$

$$F = W/3 = (400)/(3) = 133.3\ \text{N} \qquad \text{(a)}$$

Step 6: Check the solution and interpret the results. The mathematics involved are trivial. What is important is the realization that this pulley/cable arrangement permits a weight of 133.3 N to be lifted person to lift a weight of 400 N with a force of only 133.3 N. This pulley/cable arrangement is a machine enabling one to amplify applied forces. While the forces are amplified, the work to lift the weight remains constant. The forces are decreased by a factor of three but the distance through which the force is moved to lift the weight is increased by that same factor. No work is gained by a pulley/cable arrangement.

EXAMPLE 2.17

Consider the pulley arrangement that supports blocks A, B and C, as shown in Fig. E2.17. Blocks A and B each have a mass of 120 kg. The angle $\beta = 35°$. Determine the angle θ and the mass of block C, if the system is in equilibrium.

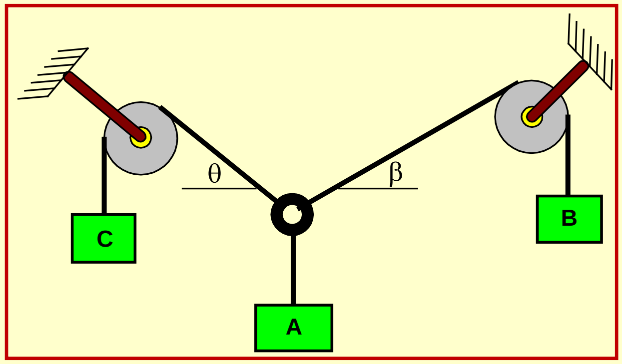


Fig. E2.17

Solution:

Prepare a FBD of the ring, as shown in Fig. E2.17a.

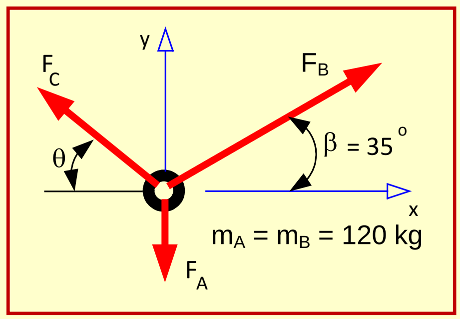


Fig. E2.17a

An examination of the FBD indicates that the force system is coplanar and concurrent. Accordingly, we may write:

$$\Sigma F_x = 0 \Rightarrow \quad F_B \cos(35°) - F_C \cos\theta = 0 \qquad \text{(a)}$$

$$\Sigma F_y = 0 \Rightarrow \quad F_B \sin(35°) + F_C \sin\theta - F_A = 0 \qquad \text{(b)}$$

Because we know the mass $m_A = m_B = 120$ kg for blocks A and B, we may write:

$$F_A = F_B = m_A\, g \qquad \text{(c)}$$

Substitute Eq. (c) into Eq. (b) and solve for F_C:

$$F_C = [(1)/(\sin\theta)][F_A - (0.5736)F_B] = (m_A g/\sin\theta)(0.4264) \qquad \text{(d)}$$

Substitute Eq. (d) into Eq. (a) and solve for the angle θ:

$$m_A g \cos(35°) - (m_A g/\sin\theta)(0.4264)\cos\theta = 0 \quad \Rightarrow \quad \tan\theta = 1/(1.921) = 0.5206$$

$$\theta = 27.50° \qquad \text{(e)}$$

Finally, to determine the mass of block C, substitute Eq. (e) into Eq. (d):

$$F_C = (m_A\, g/\sin\theta)(0.4264) = m_C\, g \quad \Rightarrow \quad m_C = (120)(0.4264)/(0.4617) = 110.8 \text{ kg} \qquad \text{(f)}$$

2.6 FORCES IN SPRINGS

Helical springs are often used to develop forces and to store energy in mechanical systems. These springs are usually wound from wire to form a helix, as shown in Fig. 2.14. This spring develops a tension force, because the hooks at both ends enable the spring to be stretched.

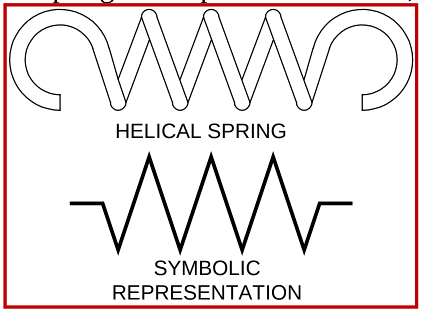


Fig. 2.14 A tension type helical spring.

If the spring is manufactured with flat ends, it is employed to support compression forces that squeeze the coils together. The stiffness or rigidity of the spring is controlled by the diameter of the wire used in winding the spring, the number of its coils and the diameter of its helix. The stiffness of the spring is known as the **spring rate** k. The force F required to elongate a tension spring or to squeeze a compression spring is given by:

$$F = k\delta \qquad (2.3)$$

where δ is the amount of extension or compression of the helical spring.

The extension (or compression) δ of the spring is established by:

$$\delta = L_d - L_o \qquad (2.4)$$

where L_d and L_o are deformed and original length of the spring, respectively.

Because drawing helical springs is time consuming and difficult, we often represent them with a zigzag line drawing as, illustrated in Fig. 2.14.

EXAMPLE 2.18

Suppose a tension spring with an original length L_o = 10.0 in. is extended to a deformed length L_d = 10.8 in. Determine the tension force required to stretch the spring to its deformed length if the spring rate is 22 lb/in.

Solution:

From Eq. (2.4), we may write:

$$\delta = L_d - L_o = 10.8 - 10.0 = 0.8 \text{ in.} \qquad \text{(a)}$$

Substitute the results from Eq. (a) into Eq. (2.3) to obtain:

$$F = k\delta = (22)(0.8) = 17.6 \text{ lb} \qquad \text{(b)}$$

EXAMPLE 2.19

Suppose a compression spring with an original length L_o = 320 mm is compacted to a deformed length L_d = 280 mm. Determine the compression force required to compress the spring to its deformed length if the spring rate is 90 N/mm.

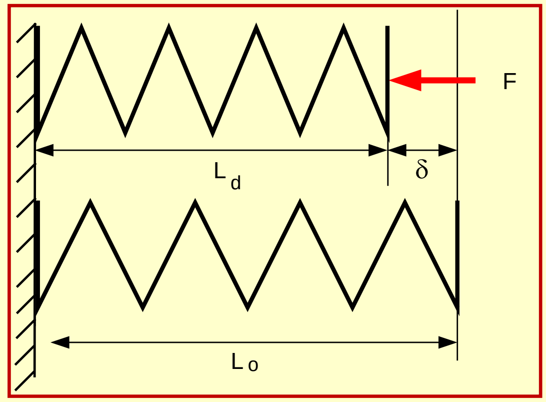


Fig. E2.19

Solution:

From Eq. (2.4), we may write:

$$\delta = L_d - L_o = 280 - 320 = -40 \text{ mm} \quad \text{(a)}$$

Substitute the results from Eq. (a) into Eq. (2.3) to obtain:

$$F = k\delta = (90)(-40) = -3.60 \text{ kN} \quad \text{(b)}$$

Note the negative sign indicates that the force acting on the spring is compressive.

2.7 MODELING PARTIAL BODIES

We are very interested in determining internal forces in structural members, because these internal forces produce the stresses and deflections that might cause the structure to fail. Consequently, determining these internal forces is an important task when designing structures. The approach followed in determining internal forces is to:

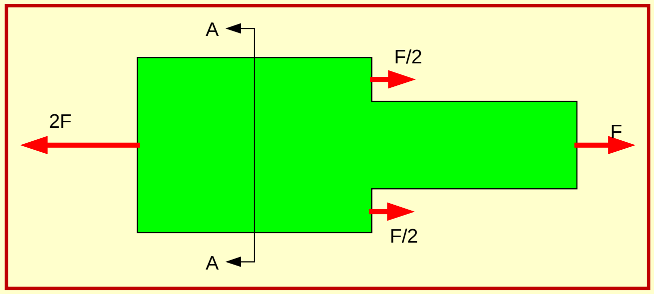
1. Make an imaginary section cut or cuts through the structural member being analyzed.
2. Draw a FBD of one or more parts of this member.
3. Account for the effect of the cut away portion of the member by applying internal forces and/or moments.
4. Solve for the internal forces or moments using the appropriate equations of equilibrium.

This procedure for constructing FBDs of partial bodies is demonstrated in Example 2.20.

EXAMPLE 2.20

Consider the stepped tension bar loaded with external forces shown in Fig. E2.20. Draw a FBD of the portion of the bar to the left side of section cut A-A.

Fig. E2.20



Solution:

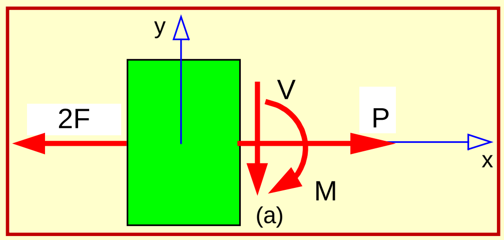
Step 1: Isolate the partial body, which is to the left of the section cut A-A, as shown in Fig. E2.20a.

Step 2: Replace the force 2F on the left part of the partial body, as shown in Fig. E2.20b.

Step 3: Place an internal force P on the right face of the partial body to account for the right side of the tension bar that has been cut away, as shown in Fig. E2.20c.

Step 4: Establish a coordinate system. In this case, dimensions are not necessary because the forces are **collinear** and **concurrent**. The complete FBD of the left portion of the tension member is shown below in Fig. 2.20a.

Fig. E2.20a



EXAMPLE 2.21

Prepare a FBD of the right portion of the tension bar, shown in Fig. E2.20, from section A-A to the bar's right hand end.

Solution:

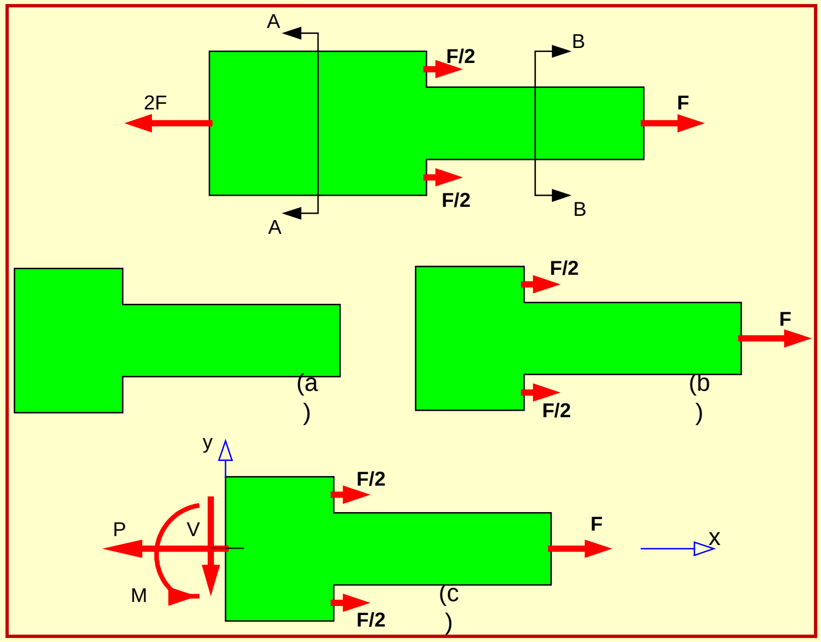
Step 1: Isolate the right portion of the bar, as shown in Fig. E2.21a.

Step 2: Replace the force F and the two forces F/2 on the right part of the body, as shown in Fig. E2.21b.

Step 3: Place an internal force P on the left face of the body to account for the left end of the tension bar that has been cut away, as shown in Fig. E2.21c.

Step 4: Establish a coordinate system. Because the forces are symmetric about the x-axis, dimensions are not necessary. The complete FBD of the right hand portion of the tension member is shown below in Fig. E2.21c.

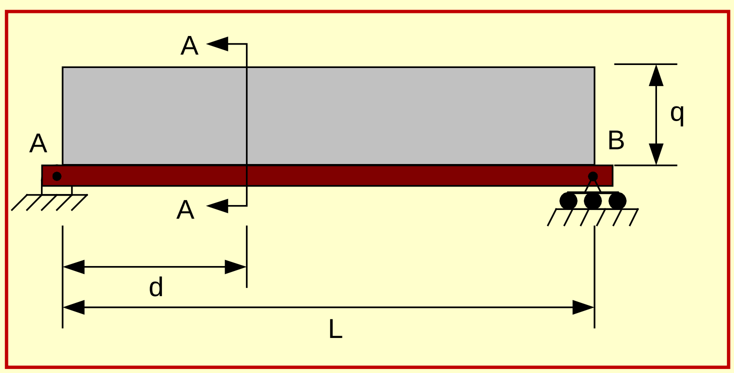
Fig. E2.21 a-c



EXAMPLE 2.22

Prepare a FBD of the left hand portion of a simply supported beam subjected to a uniformly distributed load, as shown in Fig. E2.22.

Fig. E2.22



Solution:

Step 1:. Isolate the segment of the beam to the left of section A-A, as shown in Fig. E2.22a.

Step 2: Replace the uniformly distributed load q with a concentrated force F = qd positioned at d/2 from the left end of the beam.

Step 3: Place reaction forces R_{Ax} and R_{Ay} to account for the effect of the pin-and-clevis support. Place internal forces P and V and an internal moment M_z on the exposed face to account for the effect of the right end of the beam that has been cut away. We apply a positive force P (in the positive x direction) and a positive moment M_z (counterclockwise). The force V is applied in the negative y direction. The reason for this choice for the direction of the shear force V will be explained later in the text.

Step 4: Dimension the FBD and establish a coordinate system. In analyzing beams, we usually place the origin of the coordinate system at the beam's left end. The complete FBD of the left hand portion of the beam is shown in Fig. E2.22a.

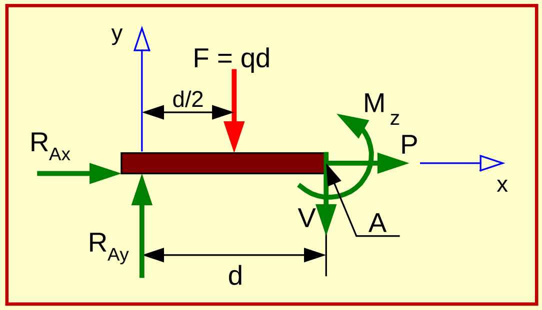


Fig. E2.22a

EXAMPLE 2.23

A new factory for heavy machinery has an assembly line with component parts stored above floor level. The components used in assembly are in elevated bins (#1, #2, and #3), as shown in Fig. E2.23. The elevated bins are suspended from a frame ABCD. Prepare a FBD of the left portion of the frame from Section A-A to the roller support at point A.

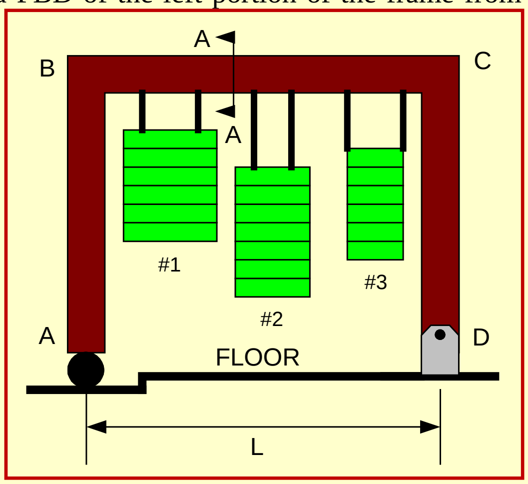


Fig. E2.23

Solution:

Step 1: Isolate the segment of the frame to the left of section A-A, as shown in Fig. E2.23a.

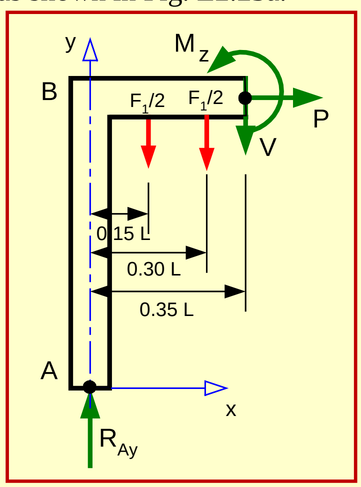


Fig. E2.23a

Step 2: Replace the loads due to the component parts on elevated bin #1 with concentrated forces $F_1/2$ and $F_1/2$ positioned at the location of the cables supporting this bin.

Step 3: Place a reaction force R_{Ay} at point A to account for the effect of the roller support. Place internal forces P and V and an internal moment M_z on the exposed face to account for the right hand portion of the frame that has been cut away. We apply a positive force P (in the positive x direction) and a positive moment M_z (counterclockwise). The shear force V is applied in the negative y direction. The reason for this choice for the direction of V will be explained later.

Step 4: Dimension the FBD and establish a coordinate system. We have placed the origin of the coordinate system at point A and have scaled the dimensions locating the cables for bin #1 from measurements of the cable locations in Fig. E2.23. The complete FBD of the left hand portion of the frame is shown in Fig. E2.23a.

2.8 SOLVING FOR INTERNAL FORCES

We now have learned to use FBDs and the equations of equilibrium to solve for external forces such as cable tension and reactions at the supports of beams and other structures. Let's now consider using the equilibrium relations to solve for the internal forces and moments that develop within a structural member due to the application of external forces and moments.

EXAMPLE 2.24

Consider the stepped tension bar loaded with external forces illustrated in Fig. E2.24.

(a) Determine the internal force P that occurs in the thick portion of the stepped bar.

(b) Determine the internal force P that occurs in the thin portion of the stepped bar.

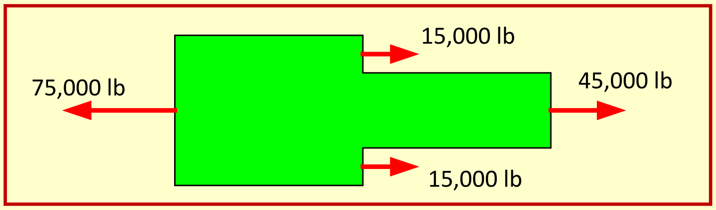


Fig. E2.24

Solution:

First, let's execute the six-step procedure to solve for the unknown magnitude of the internal force P in the thicker portion of the stepped tension bar.

Step 1: Prepare an appropriate FBD of the bar, as shown in Fig. E2.24a. We begin this FBD by making a section cut A-A, which frees the left end of the bar. We will use the left end of the bar, with the equilibrium relations, to determine the internal force P. The FBD of the left end of the bar is also presented in Fig. E2.24a.

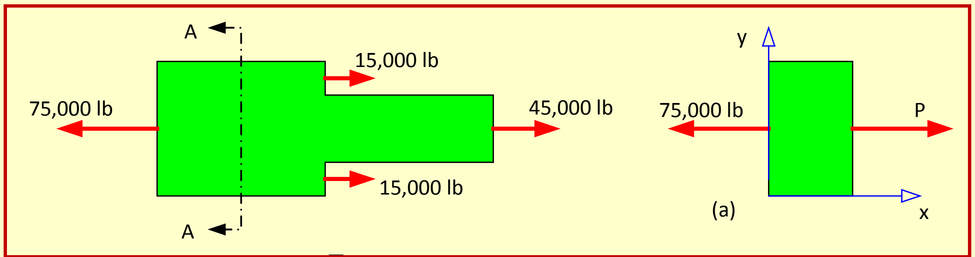
Step 2: Classify the force system. In this example, the force system is coplanar and concurrent.

Step 3: Write the relevant equations of equilibrium. In this case, only one of the six equations of equilibrium is relevant.

$$\Sigma F_x = 0$$

It is evident from inspection that the relations $\Sigma F_y = 0$ and $\Sigma M = 0$ are satisfied regardless of the value determined for the unknown force P.

Fig. E2.24a.



F

Step 4: Substitute the unknowns from the FBD into the relevant equations of equilibrium.

$$\Sigma F_x = P - 75{,}000 \text{ lb} = 0$$

Step 5: Execute the solution, which is obvious in this example.

$$P = 75{,}000 \text{ lb}$$

Step 6: Check the solution. In this problem the equations were so easy that they were solved by inspection. We have included the units in the result. Because the bar is in tension, we understand that the internal force must be positive, as indicated in our solution.

Making a section cut across the thinner end of the bar and repeating the six-step process described above solves part (b) of this example. The result obtained is P = 45,000 lb over the length of the thinner portion of the stepped tension bar. A force diagram, showing the value of P over the length of the stepped tension bar, is shown in Fig. E2.24b.

We have now explored the **solution space** for this problem, and have plotted all of the possible values of P occurring in the tension bar in the force-position diagram. By preparing a graph of the solution for each position along the length of the bar, it is possible to determine the locations where the critical forces will occur. It will be these locations that are of concern when designing structures to resist failure.

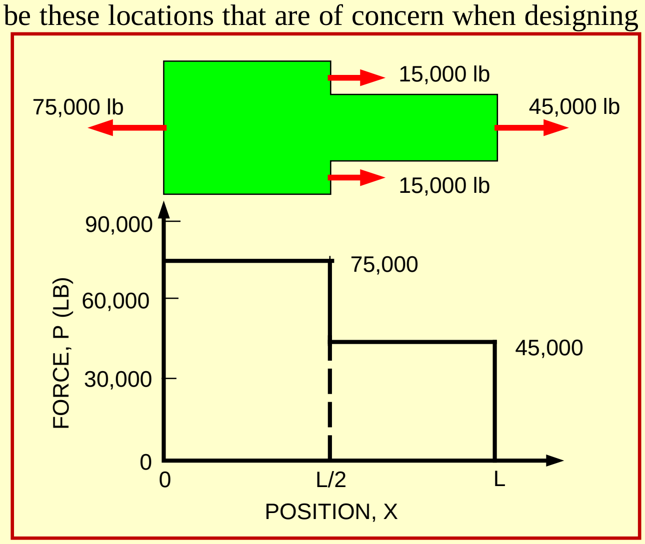


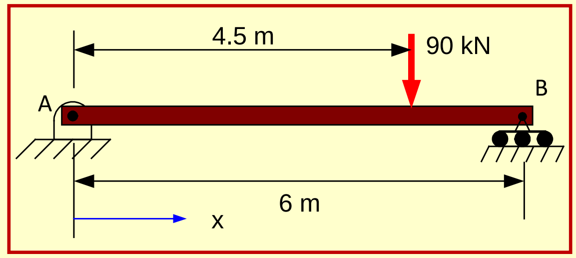
Fig. E2.24b

This example has shown that it is very easy to determine the internal forces in a tension bar. Let's consider a more complex internal force problem — a simply supported beam with a transverse concentrated force that is shown in Example 2.25.

EXAMPLE 2.25

Determine the internal forces and internal moments developed at locations defined by x = 2 m, x = 4.5 m and x = 5.0 m in the simply supported beam shown in Fig. E2.25. The beam is subjected to a transverse concentrated load of 90 kN applied at x = 4.5 m.

Fig. E2.25



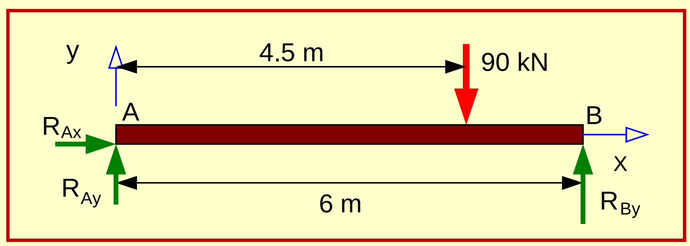
Solution:

Let us begin the solution by solving for the reactions R_{Ax}, R_{Ay} and R_{By} due to the simple supports. We construct the FBD for the entire beam following the procedure described in previous examples.

- Remove the supports.
- Add the reaction forces R_{Ax}, R_{Ay} and R_{By}.
- Define a coordinate system and dimension the FBD.

The FBD for the beam is shown in Fig. E2.25a.

Fig. E2.25a



Inspection of this FBD indicates that the force system is coplanar and non-concurrent. Hence the equations of equilibrium that apply are:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M = 0$$

The relation $\Sigma F_x = 0$ enables us to write $R_{Ax} = 0$. The relation $\Sigma M = 0$ is very useful because we are free to choose the point about which the moments are to be determined. Let's select point B at the right end of the beam. Then we write:

$$\Sigma M_B = (90\text{ kN})(1.5\text{ m}) - (6\text{ m})R_{Ay} = 0 \qquad \text{(a)}$$

$$R_{Ay} = [(90\text{ kN})/(6\text{ m})](1.5\text{ m}) = 22.5\text{ kN} \qquad \text{(b)}$$

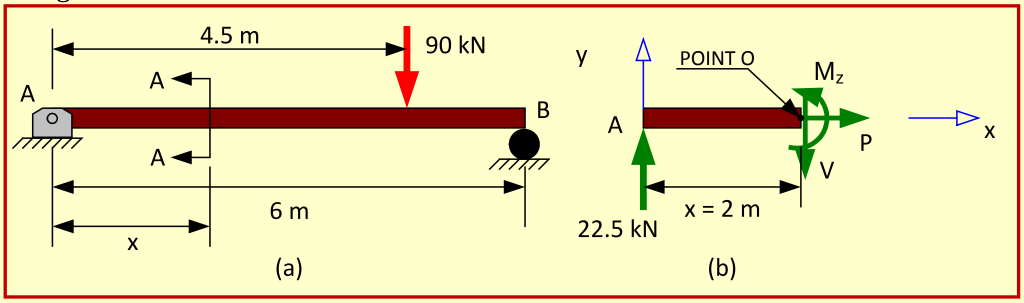
To find the reaction R_{By}, we use the remaining equation of equilibrium.

$$\Sigma F_y = R_{Ay} + R_{By} - 90 \text{ kN} = 0 \qquad \text{(c)}$$

$$R_{By} = 90 \text{ kN} - 22.5 \text{ kN} = 67.5 \text{ kN} \qquad \text{(d)}$$

Now that the external reaction forces acting at the beam supports have been determined, we proceed to solve for the internal forces at x = 2 m by making a section cut A-A through the beam at this location (see Fig. E2.25b). We then remove the left portion of the beam, freed by the section cut, as a free body. The FBD, which is the model for the equilibrium analysis, is shown in inset (b) of Fig. E2.25b.

Fig. E2.25b



The FBD shows an internal force P (tensile) in the positive x direction, and another internal shear force V, which we will consider positive when it acts in the negative y direction. An internal moment M_z about the point O is shown indicating rotation in the counterclockwise direction. We have drawn the arrows representing P, V and M_z assuming they are all positive quantities.

An examination indicates the force system is coplanar and non-concurrent. The appropriate equations of equilibrium are:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_o = 0$$

From the relation $\Sigma F_x = 0$, we find:

$$\Sigma F_x = P = 0 \qquad \text{(e)}$$

It is clear that the internal axial force P in a transversely loaded beam vanishes. From $\Sigma F_y = 0$, we find:

$$\Sigma F_y = 22.5 \text{ kN} - V = 0 \quad \Rightarrow \quad V = +\,22.5 \text{ kN} \qquad \text{(f)}$$

The positive sign for V indicates our assumption was correct — this vertical force is in the negative y direction. The internal force V is due to shear stresses distributed over the surface exposed by the section cut. We will discuss these shear stresses in much more detail in our study of Mechanics of Materials.

The relation $\Sigma M_O = 0$ permits us to solve for the internal moment M_z at the section cut A-A where x = 2m.

$$\Sigma M_O = M_z - (22.5 \text{ kN})(2 \text{ m}) = 45.0 \text{ kN-m} \qquad \text{(g)}$$

In reality, this internal moment is produced by a linear distribution of bending stresses that act normal to the surface exposed by the section cut A-A.

Let's continue our analysis of the internal forces in the beam by considering the location x = 4.5 m. Note that we freed the left portion of the beam with the section cut B-B. The cut was at x = 4.5 m, but incrementally to the left side of the 90 kN concentrated force, as shown in insert (a) of Fig. E2.25c. Accordingly, the 90 kN force is not included in the FBD shown in inset (b).

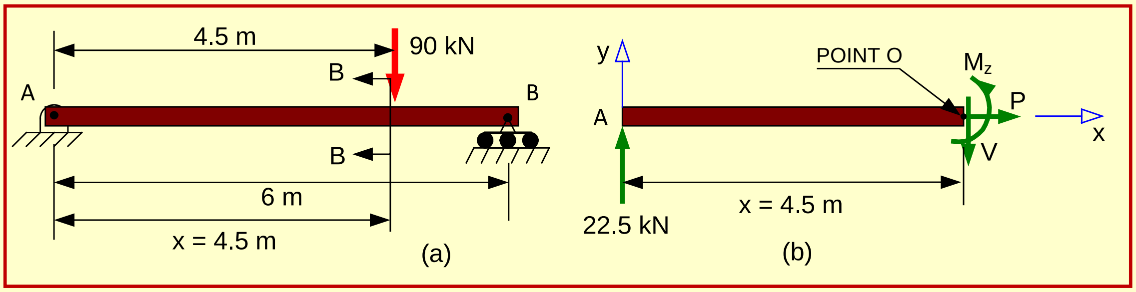


Fig E2.25c

An examination indicates the force system is coplanar and non-concurrent. The appropriate equations of equilibrium are:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_o = 0$$

From the relation $\Sigma F_x = 0$ we again find:

$$\Sigma F_x = P = 0 \qquad \text{(h)}$$

From the relation $\Sigma F_y = 0$ we find:

$$\Sigma F_y = 22.5 \text{ kN} - V = 0 \quad \Rightarrow \quad V = +\,22.5 \text{ kN} \qquad \text{(i)}$$

This result is identical to that determined for V at x = 2 m. Indeed, the shear force V is a constant for the region of the beam 0 < x < 4.5 m.

The relation $\Sigma M_O = 0$ permits us to solve for the internal moment M_z at the section cut B-B where x was slightly less than 4.5 m.

$$\Sigma M_O = M_z - (22.5 \text{ kN})(4.5 \text{ m}) = 0$$

$$M_z = 101.25 \text{ kN-m} \qquad \text{(j)}$$

If we examine the results for M_z at x = 2 and 4.5 m, it is clear that the moment increases linearly with x. Indeed, we can write an equation showing the variation of the moment with location x as:

$$M_z = 22.5\,x \qquad \text{for } 0 < x < 4.5 \qquad \text{(k)}$$

The moment M_z is in units of kN-m when x is measured in meters. We must constrain this relation for M_z to the range of x covered by the FBDs that have been considered in the analysis. At this stage in the solution, it is clear that the moment increases from zero at the left end of the beam to a maximum at x = 4.5 m. To determine the internal forces and moments in the beam for values of x > 4.5 m, it is necessary to consider a third FBD of the beam drawn with a section cut C-C located at x = 5 m, as presented in Fig. 2.25d.

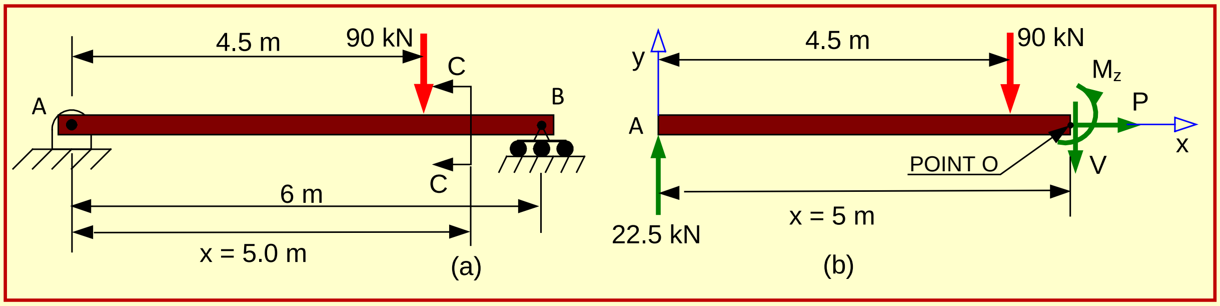


Fig. E2.25d

Following the procedure established previously, we again find that $\Sigma F_x = P = 0$

From the relation $\Sigma F_y = 0$, we determine V as:

$$\Sigma F_y = 22.5 \text{ kN} - V - 90 \text{ kN} = 0$$

$$V = -67.5 \text{ kN} \qquad \text{(l)}$$

Observe that the sign for the shear force is negative. This fact indicates that our assumption for V oriented in the negative y direction was not correct. For x > 4.5 m, the shear force is directed upwards in the positive y direction. The shear force V undergoes a step change when x locates a point incrementally to the right of the concentrated load of 90 kN.

The relation $\Sigma M_O = 0$ permits us to solve for the internal moment M_z at the section cut C-C where x = 5 m.

$$\Sigma M_O = M_z - (22.5 \text{ kN})(5 \text{ m}) + (90 \text{ kN})(0.5 \text{ m}) = 0$$

$$M_z = 67.5 \text{ kN-m} \qquad \text{(m)}$$

It is interesting to observe that the value of M_z at x = 5 m is less than its value at x = 4.5 m. If we let x = 6 m, it is easy to show that $M_z = 0$. At both supports M = 0, because the beam's simple supports cannot provide constraint against rotation and the moments must vanish at their locations.

In Example 2.25, we examined the internal forces and moments in the transversely loaded beam at three different positions along its length. We also observed that the internal moments vanish at the ends of the beam. Let's use this information to draw the diagrams presented in Fig. 2.15 showing the shear force V and the internal moment M as a function of position along the length of the beam.

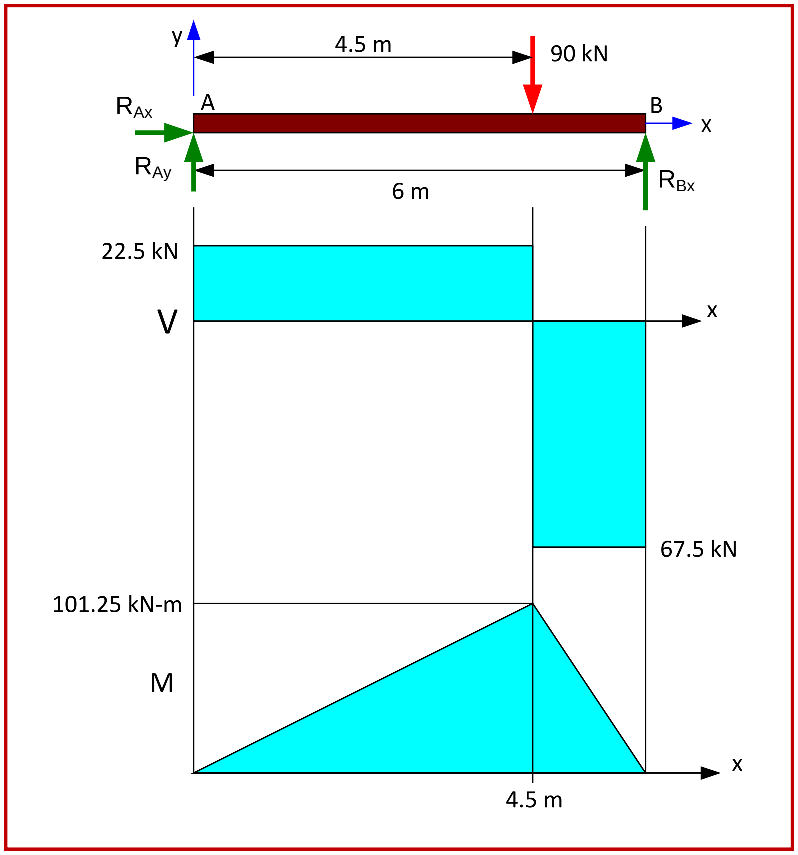


Fig 2.15 Shear and bending moment diagrams for a simply supported beam with a single transverse concentrate force at x = 4.5 m.

We will describe shear and bending moment diagrams in much more detail in our study of Mechanics of Materials, when considering bending and shear stresses produced by transverse loads applied to beams.

EXAMPLE 2.26

Determine the internal forces and moments in the uniformly loaded beam presented in Fig. E2.26. Begin by noting the symmetry of the beam's loading and the type of its supports, which enable you to solve for the reactions at the supports, as $R_{Ax} = 0$ and $R_{Ay} = R_{By} = qL/2$.

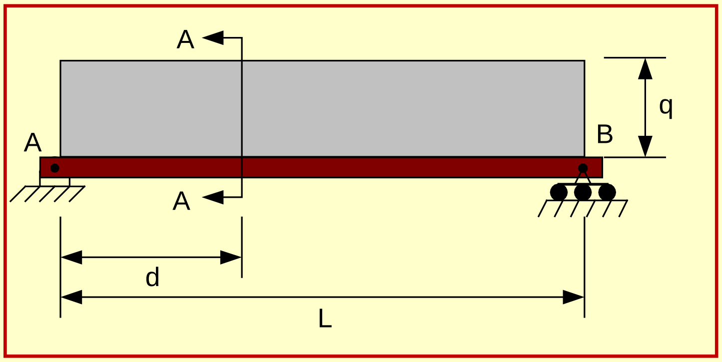


Fig. E2.26

Solution:

Let's first prepare a FBD of the left portion of the beam, as shown in Fig. E2.26a.

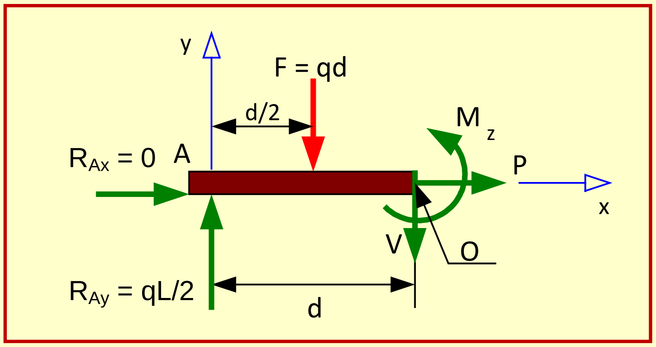


Fig. E2.26a

Examination of Fig. E2.26a, shows that the force system is coplanar and non-concurrent. Hence, we write three equilibrium equations to solve for the three unknowns P, V and M_z.

$$\Sigma F_x = R_{Ax} + P = 0 \quad \Rightarrow \quad P = -R_{Ax} = 0 \qquad \text{(a)}$$

$$\Sigma F_y = R_{Ay} - qd - V = 0 \quad \Rightarrow \quad V = qL/2 - qd = q(L/2 - d) \qquad \text{(b)}$$

$$\Sigma M_O = M_z - R_{Ay}\, d + (Fd)/2 = 0 \quad \Rightarrow \quad M_z = (qL/2)d - (qd)(d/2)$$

$$M_z = (qL/2)d - (qd)(d/2) = (qd/2)(L - d) \qquad \text{(c)}$$

The results show that the internal force V and the internal moment M_z depend on the dimension d that locates the section cut. V is a maximum when d = 0 or d = L, and M_z is a maximum when d = L/2. You may wish to construct a graph of V or M as a function of d as it ranges from 0 to L, to show the variation of the internal forces and moment over the beam's length.

EXAMPLE 2.27

Determine the internal forces and moment at section A-A in the frame of the factory building described previously in Example 2.23. Note that L = 150 ft, R_{Ay} = 150 tons, and F_1 = 60 tons.

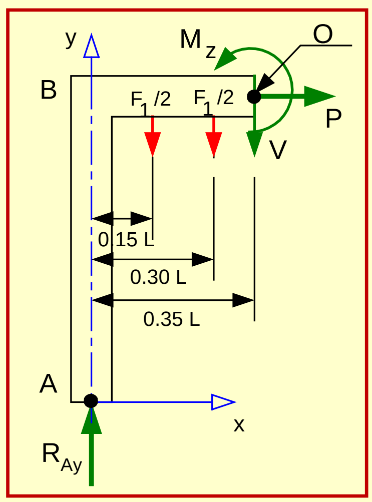


Fig. E2.27

Solution:

Examination of the FBD presented in Fig. E2.27, shows that the force system is coplanar and non-concurrent. Accordingly, we may write three equilibrium equations to solve for the three unknowns P, V and M_z.

$$\Sigma F_x = P = 0 \tag{a}$$

$$\Sigma F_y = R_{Ay} - F_1/2 - F_1/2 - V = 0$$

$$V = (150)(2{,}000) - (60)(2{,}000) = 180{,}000 \text{ lb} \tag{b}$$

$$\Sigma M_O = M_z - R_{Ay}\,(0.35L) + (F_1/2)(0.05L + 0.20L) = 0$$

$$M_z = (2{,}000)(150)[(150)(0.35) - (60/2)(0.25)] = 13.50 \times 10^6 \text{ ft-lb} \tag{c}$$

Notice that the internal force V and the internal moment M_z depend on the position of the section cut.

EXAMPLE 2.28

The C-clamp shown in Fig. E2.28 is tightened onto a block. If the screw applies a force F = 600 lb to the block, determine the internal forces and moment at the section cut A-A along the straight portion of its back.

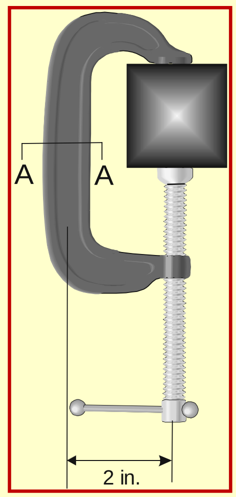


Fig. E2.28

Solution:

Let's construct a FBD of the top portion of the C-clamp, as shown in Fig. E2.28a.

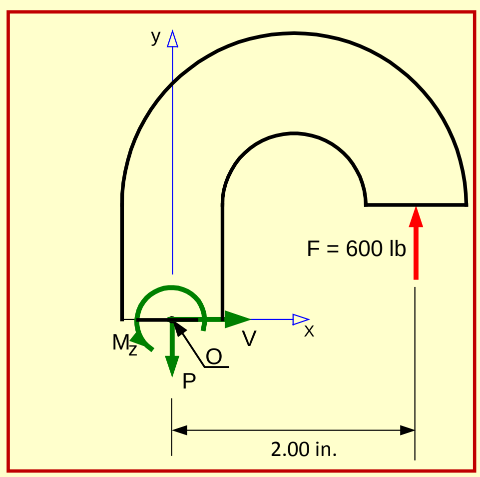


Fig. E2.28a.

Next classify the force system as coplanar, non-concurrent. Accordingly, we have three relevant equations of equilibrium. Writing these relations yields:

$$\Sigma F_x = V = 0 \qquad \text{(a)}$$

$$\Sigma F_y = F - P = 0 \Rightarrow\Rightarrow \qquad P = F = 600 \text{ lb} \qquad \text{(b)}$$

$$\Sigma M_O = (2.0)\,F + M_z = 0 \Rightarrow\Rightarrow \quad M_z = -\,(2.0)\,F = -\,1{,}200 \text{ in.-lb} \qquad \text{(c)}$$

The negative sign for M_z indicates that the direction of the rotating vector shown in Fig. E2.28a is not correct. This rotating vector should be in the clockwise direction on the FBD to maintain equilibrium of the top portion of the C-clamp.

2.9 SUMMARY

The two equilibrium equations in vector form, $\Sigma \mathbf{F} = 0$ and $\Sigma \mathbf{M} = 0$, may be represented by the six equations involving Cartesian components of the forces and moments as:

$$\boxed{\Sigma F_x = 0, \qquad \Sigma F_y = 0, \qquad \text{and } \Sigma F_z = 0 \qquad (2.1)}$$

$$\boxed{\Sigma M_x = 0, \qquad \Sigma M_y = 0, \qquad \text{and } \Sigma M_z = 0 \qquad (2.2)}$$

It is useful in selecting the appropriate equilibrium relations to classify the force system acting on a structure into one of four categories.

- Coplanar and concurrent.
- Coplanar and non-concurrent.
- Non-coplanar and concurrent.
- Non-coplanar and non-concurrent.

When the force system is classified it is apparent which of the six Cartesian equilibrium relations provide meaningful results.

It is difficult to overemphasize the importance of preparing complete and correct FBDs. The FBD represents a model of the structure under consideration. It provides a guide in writing the equilibrium equations, information on pertinent dimensions and a record of the assumptions regarding the direction of the forces and moments. A systematic method for constructing FBDs includes the following four-step procedure:

1. Isolating the body.
2. Replacing supports with reaction forces and/or moments.
3. Replacing distributed loads with concentrated loads positioned at the centroid of the area representing the distributed load.
4. Dimensioning the FBD and defining a suitable coordinate system.

Methods for solving equilibrium problems are covered in detail. A systematic six step technique is suggested that includes:

1. Prepare a complete FBD of the component under consideration.
2. Classify the force system.
3. Write the relevant equations of equilibrium.
4. Substitute the known and unknown quantities shown in the FBD into the relevant equations of equilibrium.
5. Execute the solution.
6. Check the solution and interpret the results.

Many examples are provided to demonstrate techniques for solving for reaction forces in structures with coplanar force systems. The following relevant equilibrium relations were employed depending on the classification of the force systems.

Coplanar and concurrent: $\Sigma F_x = 0 \quad \Rightarrow \quad \Sigma F_y = 0$

Coplanar and non-concurrent: $\Sigma F_x = 0 \quad \Rightarrow \quad \Sigma F_y = 0 \quad \Rightarrow \quad \Sigma M_z = 0$

The methods employed to analyze non-coplanar force systems are described in Chapter 9.

An analysis of cable/pulley arrangements was made to illustrate the ability of pulleys to change the direction of an applied force and with certain pulley arrangements to amplify an applied force. Also the helical spring was introduced as a device that develops a force in proportion to its deformation δ, and in addition stores energy. The relation governing this force as a function of displacement is given by:

$$\boxed{F = k\delta \qquad (2.3)}$$

Sectioning structural members provides a means to isolate portions of the structure. FBDs of these partial members yield models that are analyzed to solve for the internal forces and moments. Examples were provided to show techniques for determining internal forces in tension members and internal shear forces and bending moments in beams, as well as in other components and structures.

An extended example was included that introduced the concept of shear force and bending moment diagrams, which will be used extensively in a later course ENES 220, Mechanics of Materials.

CHAPTER 3

STRESS, STRAIN AND MATERIAL BEHAVIOR

3.1 INTRODUCTION

This chapter is not normally included in a Statics textbook, because usually problems in Statics involve only the determination of forces in structural elements. However, forces are rarely the only unknown quantity needed to properly design a rod, cable, column or truss. Instead it is necessary to determine the maximum stress produced by these forces and to compare that stress with the strength of the material from which the structural member is fabricated. In some cases the deformation of the structural element is required to certify that the element is sufficiently rigid. In these cases the deformation of the element is determined by using the forces imposed on the structure and the physical properties of the material from which the element is fabricated.

In this chapter, you are introduced to the process of determining stress, strain and deformation in structural elements. However, only long slender members subjected to axial forces are considered. Methods for determining stress, strain and axial deformation in these long thin members are derived. The relation between stress and strain for a uniaxial state of stress is given. The concept of shear stresses and bearing stresses with equations for their approximation are also provided. Examples are given to demonstrate the computational methods used in the solution to typical problems.

The materials from which a structural element is fabricated must be characterized in engineering terms. Critically important in this characterization is the strength and rigidity of the materials and the testing methods used to determine them. The tensile test is described in considerable detail and illustrations of the equipment employed in a standard tensile test are shown. Material properties including the fracture strength, yield strength, ultimate tensile strength, modulus of elasticity, shear modulus, Poisson's ratio and the percent elongation and percent area reduction are defined. Finally definitions of true stress and true strain are provided.

3.2 NORMAL STRESSES, STRAINS AND DEFORMATION

In a design analysis, we compute the axial forces acting on the uniaxial member, the amount the member stretches under load and the stresses developed. Next, we compare the stresses acting on the structural member, select the maximum stress, and compare it with the strength of the material from which it is fabricated. If the maximum stress exceeds the strength of the structural member, it will fail. This section describes methods that will enable you to size uniaxial members so as to provide structures that will not fail.

What happens if we pull on a wire or rod and continue to pull with increasing force? The wire or rod deforms by stretching, and continues to deform until it fails by breaking. Consider a straight scribe line drawn across the width of a piece of wire or rod, as shown in Fig. 3.1a. This line represents the edge of a plane through the cross section of the member. Next apply an axial load F to the member as indicated in Fig. 3.1b. The member will stretch a small amount δ, but the line remains straight and the plane through the cross section remains plane (i.e. flat). This is a very important observation, because it implies that the normal stress σ and the normal strain ε are uniformly distributed across this plane.

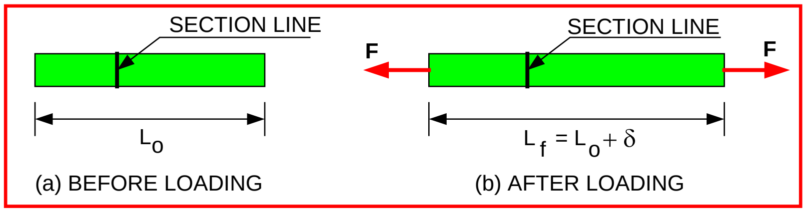


Fig. 3.1 A scribe line drawn across the member remains straight after loading.

3.2.1 Stretch of a Uniaxial Member under Load

When tensile forces are applied to a uniaxial member it stretches, as shown in Fig. 3.1b. The **deformation** of the uniaxial member δ is defined as:

$$\delta = L_f - L_o \tag{3.1}$$

where L_f is the length of the uniaxial member under load and L_o is the original length.

A FBD of a segment of the uniaxial element (a long thin rod with a uniform cross section) is shown in Fig. 3.2b. The equilibrium relation ($\Sigma F_x = 0$) indicates that the internal force P, anywhere along the length of the uniaxial member equals the external force F.

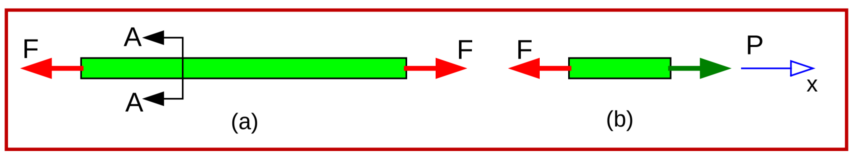


Fig. 3.2 FBD of a long thin member subject to axial load.

The amount of the deformation δ is determined from:

$$\delta = (PL)/(AE) \tag{3.2}$$

where P is the internal force in the uniaxial member, in N or lb.

L is the length of the uniaxial member, in m or in.

$A = \pi r^2$ is the cross sectional area of the uniaxial member, in m^2 or in^2.

r is the radius of the uniaxial member, in m or in.

E is the **modulus of elasticity**, in Pa or psi.

We will derive Eq. (3.2) later in this section. It is important at this stage to understand that the deformation of the uniaxial member δ is a very small quantity if the member is fabricated from a metal such as steel, copper, aluminum, etc. To demonstrate the magnitude of the deformation, consider the following example.

EXAMPLE 3.1

A No. 30 gage steel music wire (0.080 in. in diameter) is employed to lift a weight of 315 lb. If the wire is 9 ft. in length, determine the amount that the wire deforms under load.

Solution:

In Appendix B-1, we note that the modulus of elasticity for steel is listed as $E = 30 \times 10^6$ psi. In Appendix A, we verify the diameter of No. 30 gage steel music wire as 0.080 in. Next, let's substitute the values for P, L, $A = \pi r^2$ and E into Eq. (3.2) to obtain:

$$\delta = (PL)/(AE) = (315 \text{ lb})(9 \text{ ft})(12 \text{ in/ft})/[\pi\,(0.040 \text{ in.})^2\,(30 \times 10^6 \text{ lb/in}^2)]$$

$$\delta = 0.2256 \text{ in.} \qquad \text{(a)}$$

Is 0.2256 in. a small deformation for the wire? Small is a relative term and must be compared to another value to be judged. Let's compare the amount of this deformation to the original length of the wire by computing the ratio:

$$\delta/L_o = (0.2256)/(9)(12) = 0.002089 \text{ or } 0.2089\% \qquad \text{(b)}$$

The 315 lb load deforms the wire by about two tenths of one percent. It is evident that the deformation is very small when compared to the original length of the wire.

3.2.2 Strain in the Uniaxial Member under Load

Let's conduct a simple experiment[1] and observe the behavior of a uniaxial member under the action of a monotonically increasing tensile force. If we measure the force F applied to the uniaxial member, the deformation δ of the member and equate F_{EXT} to P_{INT}, we can construct the graph shown in Fig. 3.3a.

Note a linear relationship exists between P and δ as indicated by Eq. (3.2). This P-δ curve is a straight line until the uniaxial member begins to fail by **yielding**. The linear portion of the P-δ relation is the **elastic region** of the load-deflection response. Equation (3.2) is valid only in this elastic region.

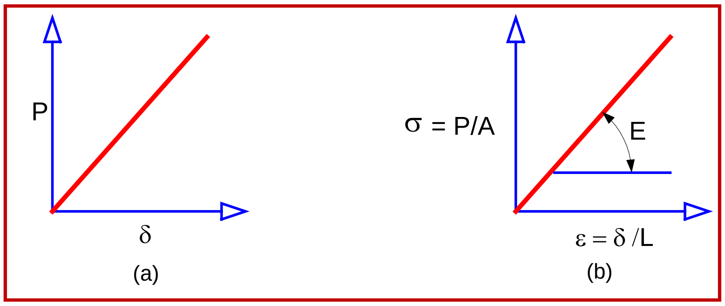


Fig. 3.3 (a) Graph of load versus deformation.
(b) Stress versus strain prior to yielding.

Let's modify the axes in Fig. 3.3a by dividing the force P by the cross sectional area A and the deformation δ by the length L as indicated below:

$$\varepsilon = \delta/L \qquad (3.3)$$

$$\sigma = P/A \qquad (3.4)$$

where ε is the strain and σ is the stress in the uniaxial member, respectively.

[1] Formal test procedures to determine material properties will be presented in Sections 3.5 and 3.6.

These relations show that the **normal strain** ε is the change in length divided by the original length — a dimensionless quantity. The **normal stress** σ is the internal force P divided by the area A over which the force acts. The stress is expressed in terms of Pa (Pascal) in SI units or psi (lb/in^2) in U.S. Customary units. We have shown a graph of stress versus strain in Fig. 3.3b. As expected there is a linear relation between stress and strain until the stress is sufficient to cause the uniaxial member to yield. The **slope** of the σ–ε line, in the elastic region, is the **modulus of elasticity** E of the material from which the uniaxial member is fabricated. The **modulus of elasticity** is a **material property**, which is independent of the shape of the body. More extensive descriptions of material properties and their methods of measurement are given later in Sections 3.5 and 3.6.

It is evident from the linear response in the stress-strain diagram that:

$$\sigma = E\,\varepsilon \qquad (3.5)$$

This stress-strain relation is known as **Hooke's law**. It is named after Robert Hooke who is credited with the discovery of elasticity in the 17th century. A word of caution — Hooke's law is valid only for uniaxial states of stress that arise in long, thin structural members. We will introduce another more complex form of the stress-strain relation to accommodate multi-axial stress fields later in the course titled Mechanics of Materials.

Let's combine Eqs. (3.3), (3.4) and (3.5) in the manner shown below:

$$\sigma = P/A = E\,\varepsilon = (E\,\delta/L) \qquad (a)$$

and solve Eq. (a) for δ to derive Eq. (3.2) as:

$$\delta = (PL)/(AE) \qquad (3.2)$$

EXAMPLE 3.2

Determine the strain that develops in a 7 m long uniaxial member, which is deformed by 8 mm.

Solution:

Using Eq. (3.3), we write:

$$\varepsilon = \delta/L = (8 \times 10^{-3}\ \text{m})/(7\ \text{m}) = 0.001143 \text{ dimensionless} \qquad (a)$$

Note that strain is dimensionless, because it is a change in length divided an original length. Also, observe that we have determined the strain without knowledge of the material used in fabricating the uniaxial member. **Strain is a geometric concept.** The strain imposed on a uniaxial member may be determined without knowledge of the internal force P, the modulus of elasticity or the identity of the material, **if** the amount of deformation and the original length of the member are known.

EXAMPLE 3.3

Determine the strain that develops in a 12 ft long uniaxial member that is subjected to a force of 120 lb. The member is fabricated from Gage 10 wire (0.10189 in. in diameter). The wire material is an aluminum alloy 2024-T5.

Solution:

Using Eq. (3.3), we write:

$$\varepsilon = \delta/L \qquad \text{(a)}$$

Recall Eq. (3.2):

$$\delta = (PL)/(AE) \qquad \text{(b)}$$

Combining Eqs. (3.2) and (3.3) yields:

$$\varepsilon = P/(AE) \qquad (3.6)$$

From Appendix B-1, we find the modulus of elasticity E for aluminum alloy 2024-T4 is 10.4×10^6 psi. Next, we compute the cross sectional area of the wire as:

$$A = \pi r^2 = \pi d^2/4 = (\pi)(0.10189)^2/(4) = 0.008154 \text{ in.} \qquad \text{(c)}$$

Substituting values for P, A and E into Eq. (3.6) gives:

$$\varepsilon = 120 \text{ lb}/[(0.008154 \text{ in.}^2)(10.4 \times 10^6 \text{ lb/in.}^2)] = 0.001415 \qquad \text{(d)}$$

In this example, it was necessary to use the value of the modulus of elasticity in determining the strain. Why? The modulus of elasticity was needed because the deformation of the wire was not specified in the problem statement. It was necessary to determine the deformation of the wire due to the force P using Eq. (3.2). This step in the analysis required us to introduce a material property — the modulus of elasticity.

3.2.3 <u>Internal Forces and Stresses</u>

Internal forces that develop within a structural member, when it is subjected to external forces, produce stresses. In Chapter 2, we reviewed the concept of internal forces and showed that they are determined from the appropriate equations of equilibrium. We also indicated the necessity for making an imaginary section cut to expose the cross sectional area of the member over which the stresses act. In this section, we emphasize the connection between the external forces, internal forces and the stresses that develop at any point along the length of the uniaxial member.

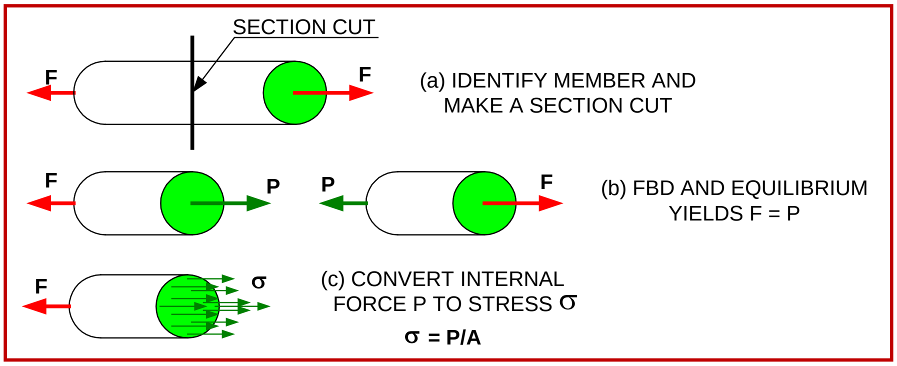


Fig. 3.4 A diagram showing the transition from external forces to internal forces to stresses.

To illustrate these connections, examine Fig 3.4, which shows an axial member subjected to an external axial force F. To determine the internal forces and the resulting normal stresses a four step procedure is employed:

1. We identify the member of interest and the forces acting on it in Fig. 3.4a.
2. A section cut is made at a location of interest along the length of the member. We then remove one end of the member to construct the FBD depicted in Fig. 3.4b.
3. The equilibrium relation $\Sigma F_x = 0$ is written proving that F (external) = P (internal).
4. Using Eq. (3.4) ($\sigma = P/A$), we convert the internal force P into the normal stress σ as indicated in Fig. 3.4c.

In Fig. 3.4, the internal force P is a constant over the length of the member and the location of the section cut is not important. However, be aware that in some applications the internal forces vary from one location to another. In these situations, the location of the section cut is vitally important. Also observe in Fig. 3.4c that the distribution of the stress σ is uniform over the cross sectional area exposed by the section cut. The uniform distribution of the stress is a result of two conditions:

- Plane sections remain plane for long, thin members subjected to axial forces.
- Internal moments do not develop because the external forces are applied along the axis of the member.

In the most general case where the normal stresses may not be uniformly distributed, we relate the internal force and the normal stress over an area A by:

$$P = \int \sigma \, dA \tag{3.7}$$

However, when the stress is uniformly distributed, as shown in Fig. 3.4, Eq. (3.7) reduces to:

$$P = \sigma \int dA = \sigma A \tag{3.8}$$

EXAMPLE 3.4

Determine the stress in a 4.0 mm diameter uniaxial member subjected to an axial load of 1,100 N.

Solution:

From Eq. (3.4), we write

$$\sigma = P/A \qquad \text{(a)}$$

From a FBD identical to the one shown in Fig. 3.4b and the equilibrium relation $\Sigma F_x = 0$, we understand that:

$$F = P = 1{,}100 \text{ N} \qquad \text{(b)}$$

The area $A = \pi d^2/4 = \pi(4.0 \text{ mm})^2/4 = 12.57 \text{ mm}^2$. Substituting these values into Eq. (a) gives:

$$\sigma = (1{,}100 \text{ N})/(12.57 \text{ mm}^2) = 87.51 \text{ N/mm}^2 = 87.51 \text{ MPa} \qquad \text{(c)}$$

Note that one N/mm^2 is equivalent to one MPa. Later in this chapter, an interpretation of this result is presented after describing in detail the strength of structural materials in Section 3.6.

EXAMPLE 3.5

A large diameter uniaxial member fabricated from steel with a cross sectional area $A = 11.46 \text{ in.}^2$ is to support a portion of the roadway on a suspension bridge. The highway engineers have specified that the maximum force imposed on the cable is not to exceed 95 ton. Determine the stress and strain in the uniaxial member when subjected to the specified load.

Solution:

Let's compute the stress from Eq. (3.4) as:

$$\sigma = P/A = (95 \text{ ton})(2{,}000 \text{ lb/ton})/(11.46 \text{ in.}^2) = 16{,}580 \text{ psi} = 16.58 \text{ ksi} \qquad \text{(a)}$$

Next, calculate the strain ε from Eq. (3.6) to obtain:

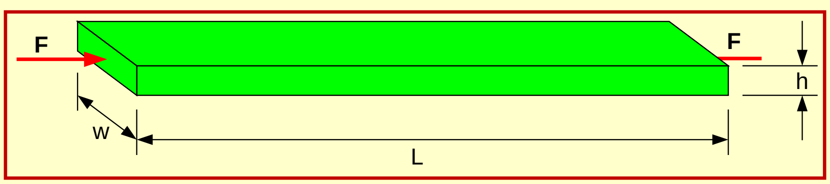
$$\varepsilon = P/(AE) = (190{,}000 \text{ lb})/[(11.46 \text{ in}^2)(30 \times 10^6 \text{ lb/in}^2)] = 0.5526 \times 10^{-3} \qquad \text{(b)}$$

In determining the strain, we obtained the value of $E = 30 \times 10^6$ psi for steel from Appendix B-1. We also introduced ksi, a unit in the U. S. Customary system. The conversion factor between psi and ksi is given by ⇒ 1 ksi = 1,000 psi.

EXAMPLE 3.6

Determine the stress in the bar shown in Fig. E3.6 when subjected to a compressive axial force of F that acts through the centroid of the bar. The following numerical parameters define the bar and the applied load: F = 700 kN, L = 900 mm, w = 125 mm and h = 80 mm.

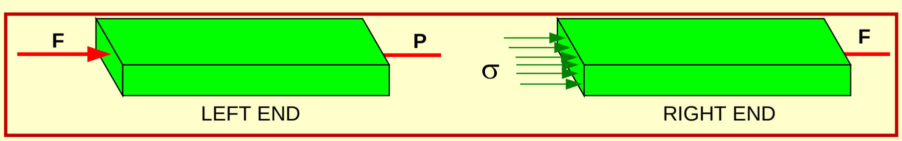
Fig. E3.6



Solution:

We make a section cut near the center of the bar, and construct the free body diagram shown in Fig. E3.6a:

Fig. E3.6a



The stresses σ visible on the face exposed by the section cut are due to the internal force P. It is clear from $\Sigma F_x = 0$ that P = F; then from Eq. (3.4), we write:

$$\sigma = P/A = (-\,700 \times 10^3 \text{ N})/[(125)(80) \text{ mm}^2]$$

$$\sigma = -\,70.00 \text{ N/mm}^2 = -\,70.00 \text{ MPa}$$

The axial stress is a negative number (– 70.00 MPa) indicating that a compressive stress develops in the bar due to the compressive axial loading. Also observe that the value of the stress is independent of the material used to fabricate the bar.

EXAMPLE 3.7

If the bar described in Example 3.6 is fabricated from steel, determine the length of the bar after the application of the compressive force.

Solution:

An inspection of the free body diagram shown in Fig E3.6a indicates that the internal force P is constant from one end of the bar to the other and equal to the external force F. Using Eqs. (3.1) and (3.2), we may write:

$$\delta = L_f - L_0 = (PL_0)/(AE) \qquad \text{(a)}$$

Solving Eq. (a) for L_f, gives:

$$L_f = L_0\,[1 + P/(AE)] \qquad \text{(b)}$$

Substituting numerical values for the known parameters in Eq. (b) yields:

$$L_f = 900\left[1 - \frac{700 \times 10^3}{(80)(125)(207 \times 10^3)}\right]$$

$$L_f = (900)[1 - 0.33816 \times 10^{-3}] \text{ m} = (900 - 0.3043) \text{ mm} = 899.6957 \text{ mm} \qquad \text{(c)}$$

Because the new length of the rod is 899.6957 mm, it is apparent that it contracted by 0.3043 mm under the action of the compressive load. The value of P in Eq. (3.2) is treated as a negative number when the axial load on the bar is compressive. Also, the value of the modulus of elasticity was taken as 207 GPa (i.e. 207,000 MPa) from Appendix B-1.

EXAMPLE 3.8

Determine the axial deformation in a 6-foot long rod that has a diameter of 3 in. The rod is fabricated from mild (low strength) steel and is subjected to an axial compressive load of 70,000 lb. Repeat the solution for the rod if it is fabricated from an aluminum alloy.

Solution:

Again, we begin with a drawing of the rod and a free body diagram in Fig. E3.8, showing the forces F and P and the normal stresses σ. An inspection of the free body diagram in Fig. E3.8 indicates that the internal force P is constant over the entire length of the rod. From Eq. (3.2), we write:

$$\delta = (PL)/(AE) \qquad \text{(a)}$$

Substituting the values for the known quantities for the steel rod in Eq. (a) gives:

$$\delta = -\,[(70{,}000)(6)(12)] \,/\, [(\pi)(1.5)^2\,(30 \times 10^6)] = -\,0.02377 \text{ in.} \qquad \text{(b)}$$

Substituting the values for the known quantities for the aluminum rod in Eq. (a) gives:

$$\delta = -\,[(70{,}000)(6)(12)] \,/\, [(\pi)(1.5)^2\,(10.4 \times 10^6)] = -\,0.06856 \text{ in.} \qquad \text{(c)}$$

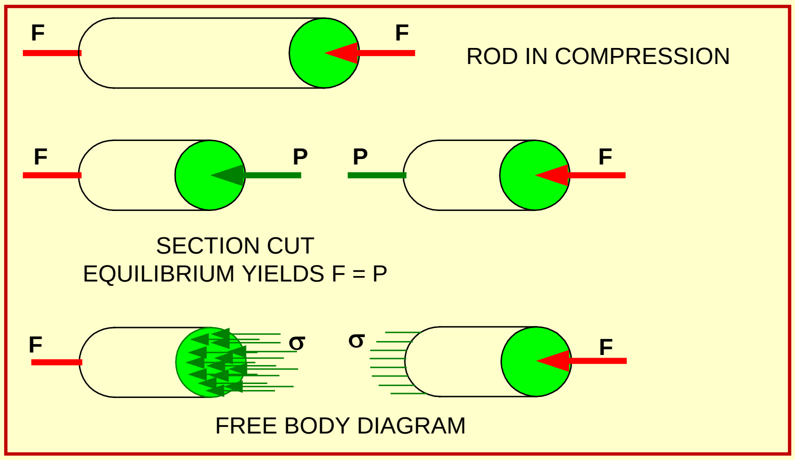


Fig. E3.8

Both solutions carry a negative sign indicating that the rod is compressed (shortened) by the action of the compressive force. It is interesting to observe that the rod fabricated from an aluminum alloy exhibited nearly three times the deformation of the steel rod. The reason for this difference is in the lower modulus of elasticity of aluminum relative to steel. The modulus of elasticity for aluminum and steel used in this calculation was 10.4×10^6 and 30×10^6 psi, respectively, as cited in Appendix B-1.

3.3 SHEAR STRESSES

In the preceding discussions, the stresses created by internal forces were in a normal direction to the area exposed by the section cut. To emphasize this fact, we called them **normal stresses**. A second type of stress exists — a **shear stress**. As the name implies, the shear stress lies in the plane of the area exposed by the section cut. To show shear stresses in a more graphical manner, consider the stubby block-like member loaded with the force F in Fig. 3.5. First, cut the stubby member to create a free body of the left end, and then apply an internal shear force V in the plane of the area exposed by the section cut[2]. From $\sum$

[2]To maintain the focus of the discussion on shear forces, we have not included the internal moment acting at the section cut on the FBD in Fig. 3.5.

F_y = 0, we determine that F = V. The shear force V produces a shear stress τ. The relation between the shear stress τ and the shear force V is:

$$\tau = V/A \tag{3.9}$$

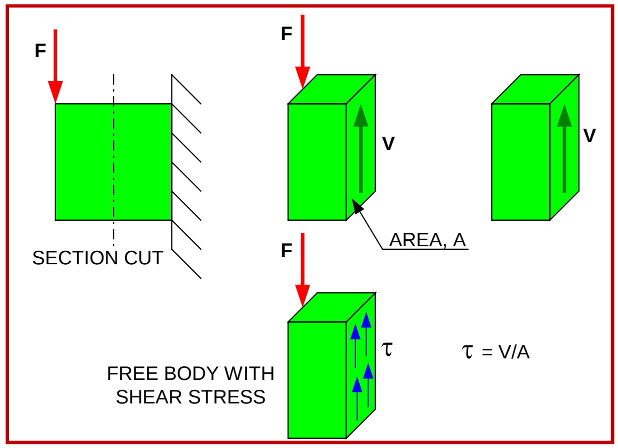


Fig. 3.5 A section cut made in a stubby block-like member produces a free body.

We have assumed that the shear stress τ is uniformly distributed over the area of the stubby member. Later, in the discussion of beam theory covered in the course on Mechanics of Materials, we will note that the shear stress is not uniformly distributed over the cross sectional area of a beam. However, for many block-like (short and stubby) members, the assumption of a uniform distribution of shear stresses is a reasonable approximation.

EXAMPLE 3.9

A key is employed to keep a gear from slipping on a shaft when transmitting power. Forces F of 50 kN are created on the key at the locations shown in Fig. E3.9. Determine the shear stresses in the key if it is 12 mm wide, 12 mm high and 75 mm long.

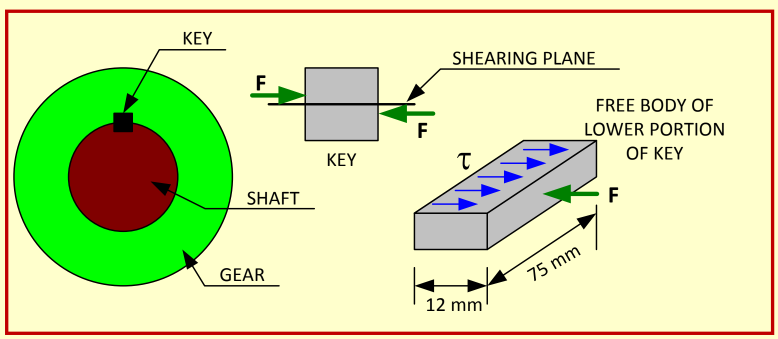


Fig. E3.9 A section cut showing shear forces acting on the key, which prevents the gear from slipping on the shaft

Solution:

In Fig. E3.9 we construct a FBD of the key showing the equal and opposite forces F, and the shear plane between the gear and the shaft. We section the key along the shear plane and draw another FBD of its lower portion. On this second FBD, we show the shear stresses τ that occur on the shear plane.

The shear stresses on the key are given by Eq. (3.9) as:

$$\tau = V/A = (50 \times 10^3)/[(12)(75)] = 55.56 \text{ N/mm}^2 = 55.56 \text{ MPa} \qquad \text{(a)}$$

3.3.1 Analysis of Joints Fabricated with Bolts and Rivets

In the design of structures and pressure vessels, joints connecting two or more members are often required. These joints can be fabricated by welding, bolting or riveting. Let's consider several joint designs fabricated with either bolts or rivets, as shown in Fig. 3.6. The joint presented in Fig. 3.6a is a lap joint with rivets having a single shear plane. The illustration in Fig. 3.6b shows a butt joint and a single strap with rivets having a single shear plane. The joint depicted in Fig. 3.6c is a butt joint and a double strap with rivets having a double shear plane.

These rivet or bolt joints are often used to connect wide plates[3] with the fasteners placed in single, double or triple rows depending on the forces that are transmitted across the joint. An example of a single row of rivets in a lap joint is presented in Fig. 3.7.

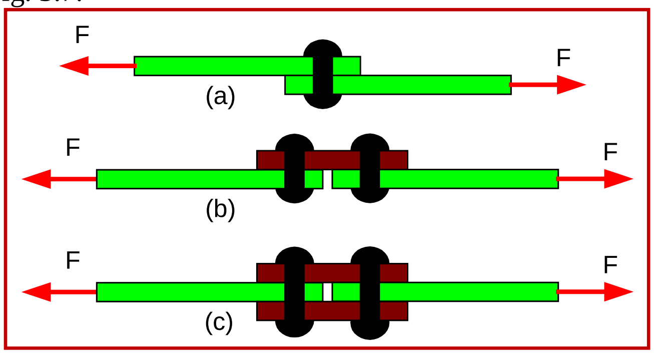


Fig. 3.6 Lap and butt joints with rivet fasteners.
(a) Lap joint
(b) Butt joint with a single strap.
(c) Butt joint with a double strap.

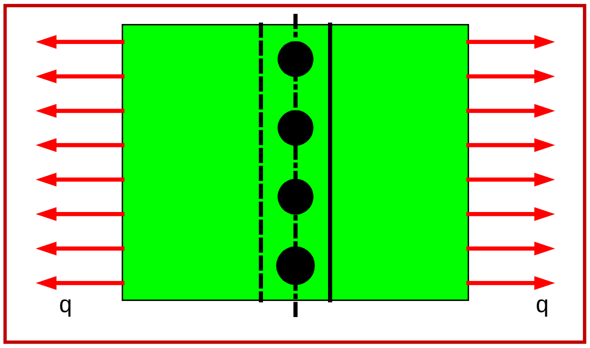


Fig. 3.7 A single row of rivets to fasten two overlapping plates forming a lap joint in a wide plate.

In these lap and butt joints, we are concerned with shear stresses that develop in the rivets and with the normal stresses developed along the section of the plate containing the rivet holes. Let's begin by drawing a FBD of the rivet in a single-shear-plane, lap joint, as shown in Fig. 3.8. Note that friction forces that occur between the plates that are fastened together have been neglected, when preparing this FBD.

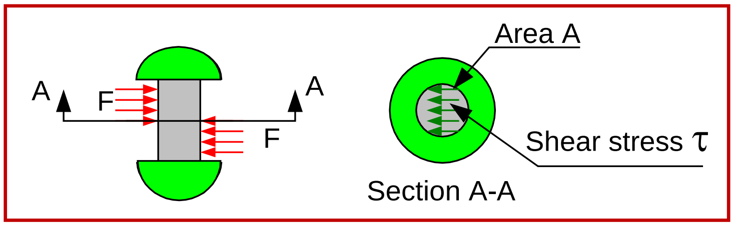


Fig. 3.8 FBD of the rivet in a single shear plane lap joint.

[3] Aircraft structures fabricated from thin plates of aluminum employ many riveted joints to form complex structures.

The shear stress τ developed at the shearing plane of the rivet is given by:

$$\tau = V/A = F/A \tag{3.10}$$

where A is the cross sectional area of the bolt or rivet and F is equal to the shear force V applied parallel to the shear plane for each rivet. Although the values are identical, remember that F and V are two distinct forces – F acts on the side of the rivet while V acts on the cross sectional area.

The force F on each rivet is determined by multiplying the distributed load q acting across the plates by the width w of the plates and dividing by the number of rivets N in the lap joint to obtain:

$$F = (qw)/N \tag{3.11}$$

where q is the applied force per unit width; w is the width of the joint and N is the number of rivets.
The analysis of the butt joint that incorporates two straps (see Fig. 3.6c) is almost identical to that shown above. The FBD, presented in Fig. 3.9, shows two shearing planes, which carry the shear stresses. An equilibrium analysis on the rivet yields V = F/2, and the shear stress on each plane is given by:

$$\tau = V/A = F/(2A) \tag{3.12}$$

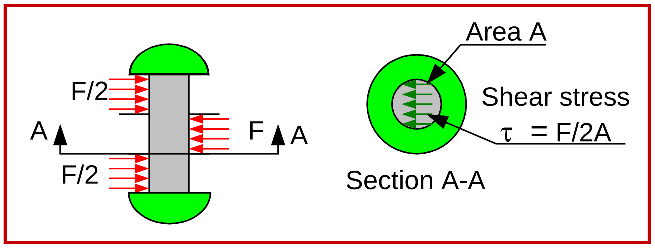


Fig. 3.9 FBD of the rivet in a butt joint with double straps.

Next consider the normal stresses σ that occur along the rivet line. Drilling holes to accommodate the rivets increases the normal stresses in the plates. To determine the elevated levels of the normal stresses along the rivet line, prepare a FBD as shown in Fig. 3.10.

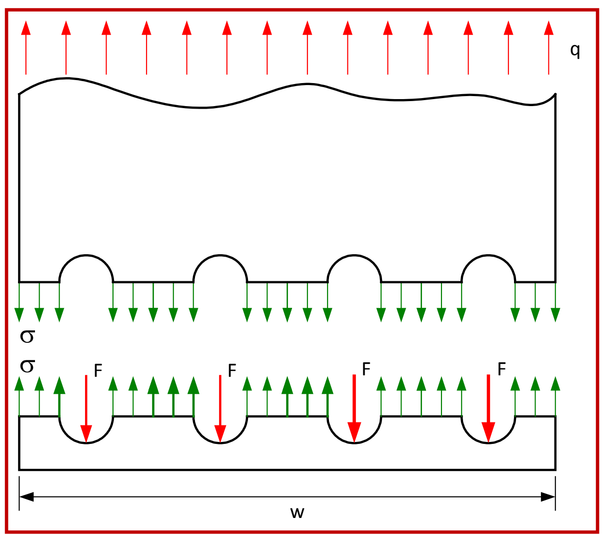


Fig. 3.10 FBD of the joint showing the normal stresses along the rivet line.

Let's consider the upper portion of the joint shown in Fig. 3.10 and write the equation of equilibrium for forces in the direction of the force per unit width q to obtain:

$$\sum F_y = qw - \sigma t(w - Nd) = 0 \tag{a}$$

where w is the width of the joint, t is the thickness of the plate, d is the diameter of the rivet hole and N is the number of rivets.

Solving Eq. (a) for σ yields:

$$\sigma = (qw)/[t(w - Nd)] \qquad (3.13)$$

EXAMPLE 3.10

The lap joint with a single shear plane, shown in Fig. E3.10, is 900 mm wide with four bolts spaced on 225 mm centers. The two plates are each 15 mm thick and the bolts are 22 mm in diameter. Determine the shear stress in the bolts if the force per unit width q is uniformly distributed over the width of the joint and equal to 110 N/mm. Also determine the normal stress acting along a section through the bolt line.

Solution:

Substituting Eq. (3.11) into Eq. (3.10) gives:

$$\tau = (qw)/(NA) \qquad (a)$$

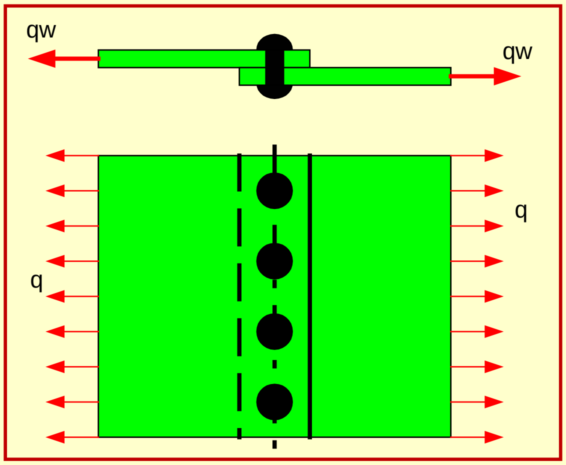


Fig. E3.10

The cross sectional area A of a single bolt is given by:

$$A = \pi d^2/4 = \pi\,(22)^2/4 = 380.1\ \text{mm}^2 \qquad (b)$$

Substituting Eq. (b) into Eq. (a) yields:

$$\tau = qw/(NA) = (110)(900)/(4)(380.1) = 65.11\ \text{MPa} \qquad (c)$$

The normal stress σ in the plate along a section through the bolt line is given by Eq. (3.13) as:

$$\sigma = (qw)/[t(w - Nd)] = [(110)(900)]/15[900 - (4)(22)] = 8.128\ \text{MPa} \qquad (d)$$

Determining the shearing stress in the bolts and the normal stress along the bolt line is the first step in the analysis of a joint. We compare these stresses to the shear and yield strengths of the materials from which the joint is fabricated to determine the safety factors.

EXAMPLE 3.11

The butt joint with double straps, shown in Fig. E3.11, is 1,200 mm wide with four bolts spaced on 300 mm centers on each bolt line. The four plates are each 25 mm thick and the bolts are 30 mm in diameter. Determine the shear stress in the bolts if the force per unit width q is uniformly distributed over the width of the joint and equal to 750 N/mm. Also determine the normal stress acting along a section through the bolt line.

Solution:

Substituting Eq. (3.11) into Eq. (3.12) gives:

$$\tau = (qw)/(2NA) \qquad \text{(a)}$$

The cross sectional area of a single bolt is given by:

$$A = \pi d^2/4 = \pi\,(30)^2/(4) = 706.86 \text{ mm}^2 \qquad \text{(b)}$$

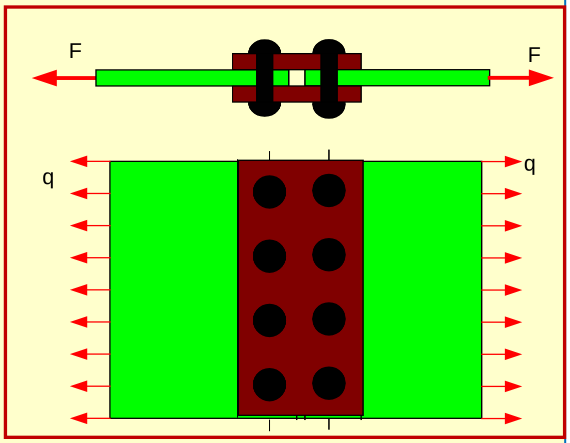


Fig. E3.11

Substituting Eq. (b) into Eq. (a) yields:

$$\tau = qw/(2NA) = [(750)(1{,}200)]/[(2)(4)(706.86)] = 159.2 \text{ MPa} \qquad \text{(c)}$$

The normal stress σ in the plate along a section through the bolt line is given by Eq. (3.13) as:

$$\sigma = (qw)/[t(w - Nd)] = [(750)(1{,}200)]/\{(25)[1{,}200 - (4)(30)]\} = 33.33 \text{ MPa} \qquad \text{(d)}$$

Determining the shearing stress in the bolts and the normal stress along the bolt line is the first step in the analysis of a joint. We compare these stresses to the shear and yield strengths of the materials from which the joint is fabricated to determine the safety factors.

3.3.2 Analysis of Pull-out in Bolted or Riveted Joints

Another mode of failure, called pullout, can occur with bolted or riveted joints. Pullout occurs when the material from the bolt line to the edge of the plate is inadequate. Let's consider a simple joint with a single rivet, as shown in Fig. 3.11. We have represented the end of the joint with a single plate that has been sectioned along the bolt line. The applied load per unit width to the top piece is q, which is balanced by normal stresses (not shown) along the section through the bolt line. The rivet applies a force F to the bottom piece, as shown in Fig 3.11. The force F is balanced by the normal stresses (not shown) along the section though the bolt line.

Next let's remove a small region of the joint below the rivet that includes the shear planes, as shown in Fig. 3.12. There is a shear stress τ that acts on each of the two shear planes, which we assume is uniformly distributed. If we consider a joint containing N rivets, the force F acting on each rivet is given by Eq. (3.11). Since there are two shearing planes, we substitute Eq. (3.11) into Eq. (3.12) to obtain the shear stress on each of the shearing planes as:

$$\tau = V/A_{Shear} = F/(2A_{Shear}) = (qw)/[2N\ (t\ s)] \qquad (3.14)$$

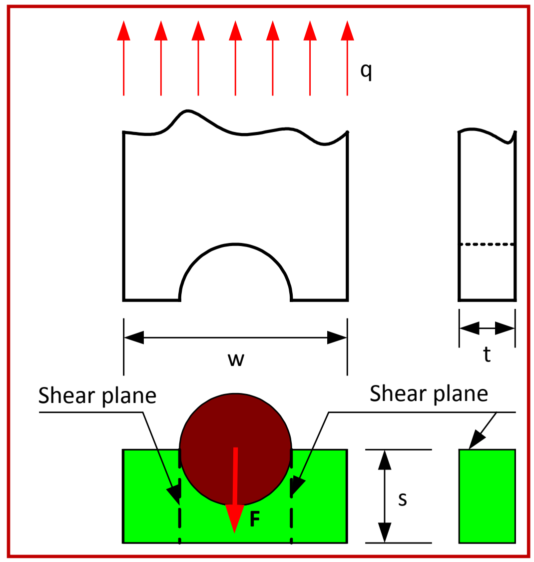


Fig. 3.11 FBD of a simple joint with a single rivet showing the shear planes associated with pullout.

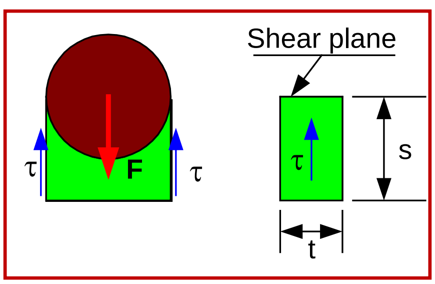


Fig. 3.12 FBD of the small region directly below the rivet that shows the shear stresses acting on the two shear planes.

EXAMPLE 3.12

For the joint described in Example 3.10, determine the shear stress associated with pullout if the distance s from the bolt centerline to the edge of the plates is 20 mm.

Solution:

The force acting on each bolt is determined from Eq. (3.11) as:

$$F = (qw)/N = (110)(900)/(4) = 24.75 \text{ kN} \qquad (a)$$

The shear stresses on each of the shearing planes is determined from Eq. (3.14) as:

$$\tau = (qw)/[2N\ (t\ s)] = [(110)(900)]/[(2)(4)(15)(20)] = 41.25 \text{ MPa} \qquad (b)$$

Determining the shearing stress that causes bolt or rivet pullout is the second step in the analysis of a joint. We compare these stresses to the shear strength of the material from which the plate is fabricated to determine its safety factor.

EXAMPLE 3.13

For the joint described in Example 3.11, determine the shear stress associated with pullout if the distance s from the bolt centerline to the edge of the plates is 30 mm.

Solution:

The force acting on each bolt is determined from Eq. (3.11) as:

$$F = (qw)/N = (750)(1{,}200)/4 = 225.0 \text{ kN} \qquad \text{(a)}$$

The shear stresses on each of the shearing planes is determined from Eq. (3.14) as:

$$\tau = (qw)/[2N\,(t\,s)] = (225 \times 10^3)/[(2)(25)(30)] = 150.0 \text{ MPa} \qquad \text{(b)}$$

Determining the shearing stress that causes bolt or rivet pullout is the second step in the analysis of a joint. We compare these stresses to the shear strength of the material from which the plate is fabricated to determine its safety factor.

3.3.3 Analysis of Lap Joints Fabricated with Adhesive

Significant advances have been made in the formulation and the application of high strength adhesives over the past two decades. Using these high strength adhesives enables engineers to design lap joints that do not contain rivets or bolts. There are significant savings with this approach, because it is not necessary to drill holes and to drive rivets or install bolts. A sketch of a lap joint fabricated with adhesives is presented in Fig. 3.13.

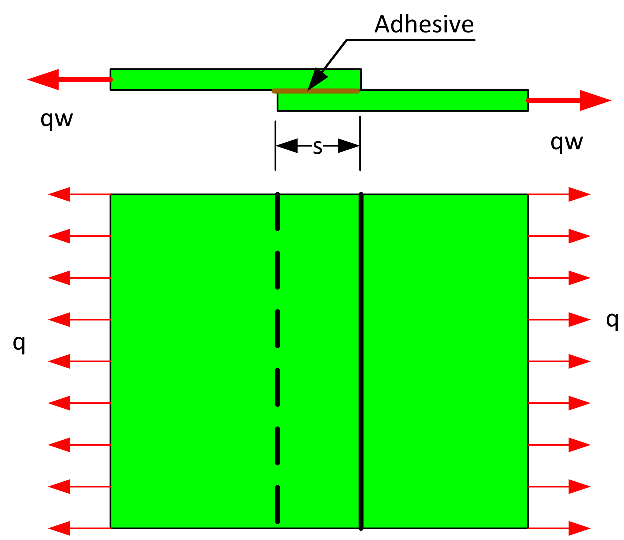


Fig. 3.13 A single lap joint fabricated with a thin film of adhesive.

The shear stress τ in the adhesive joint is given by:

$$\tau = qw/A_A = qw/ws = q/s \qquad (3.15)$$

where the area of the adhesive $A_A = ws$.

The normal stress σ on the material is given by:

$$\sigma = qw/wt = q/t \tag{3.16}$$

where t is the thickness of the sheets used in making the joint.

In some situations the shear stresses on a single lap joint become excessive and a double lap joint is employed to provide a stronger joint. The double lap joint with adhesive applied in four places is presented in Fig. 3.14.

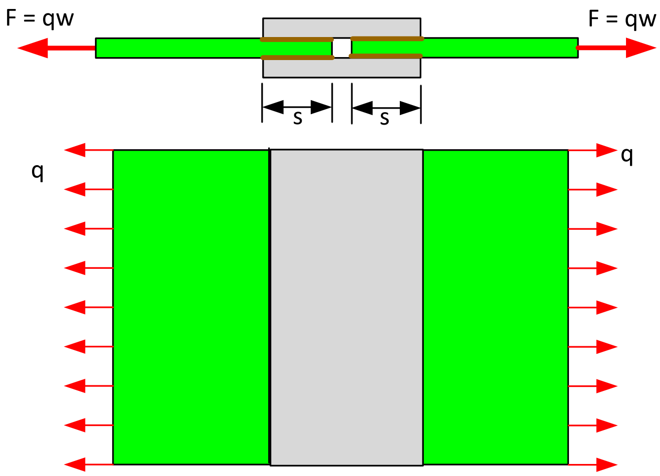


Fig. 3.14 Double lap joint with adhesive bonding.

The shear stress τ in the adhesive of this joint is given by:

$$\tau = qw/A_A = qw/2ws = q/2s \tag{3.17}$$

where the area of the adhesive A_A = ws.

The maximum normal stress σ on the material is given by:

$$\sigma = qw/wt = q/t \tag{3.18}$$

The normal stress σ on the material used in making the overlaps is given by:

$$\sigma = qw/2wt = q/2t \tag{3.19}$$

where t is the thickness of the sheet material used in making the joint.

EXAMPLE 3.14

The lap joint with a single shear plane, shown in Fig. E3.14, is 900 mm wide with an overlap 50 mm. The lap joint if fabricated using an structural adhesive with a shear strength of 20 MPa. The two plates are each 8 mm thick. Determine the shear stress in the adhesive if the force per unit width q is uniformly distributed over the width of the joint and equal to 490 N/mm. Also determine the normal stress acting on the Aluminum 6061-T6 plates used in fabricating the joint.

Solution:

Substituting into Eq. (3.15) gives:

$$\tau = q/s = 490/50 = 9.8 \text{ MPa}$$

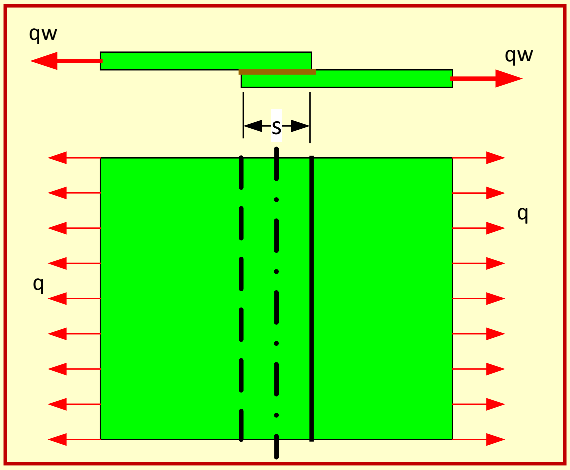


Fig. E3.14

The normal stress σ in the plate along a section removed from the overlap is given by Eq. (3.16) as:

$$\sigma = q/t = 490/8 = 61.25 \text{ MPa}$$

Inspection of the results indicate that the adhesive stress of 9.8 MPa is low compared to the 20 MPa shear strength of the structural adhesive. Also reference to a handbook of material properties indicates that the normal stress in the aluminum sheets of 61.25 MPa is small compared to the yield strength of 240 MPa for the aluminum alloy sheets.

3.4 BEARING STRESSES

Bearing stresses occur at the interface between two bodies in contact. Consider the case of a cylinder of diameter d in contact with a large-thick-flat plate, as shown in Fig. 3.15.

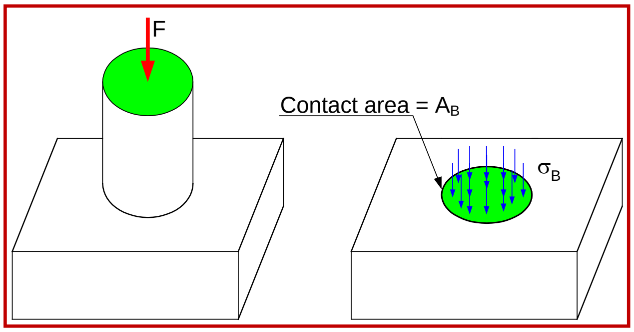


Fig. 3.15 Bearing stresses σ_B occur at the interface of two bodies in contact.

An axial force F is applied to the cylinder, which is maintained in equilibrium by the reaction at the interface between the cylinder and the thick plate. The stresses that develop at this interface are called bearing stresses denoted by the symbol σ_B. The bearing stresses are determined from:

$$\sigma_B = F/A_B \qquad (3.20)$$

where A_B is the contact (bearing) area

3.4.1 Bearing Stresses in Bolted or Riveted Joints

Determining the bearing stresses developed in a riveted or a bolted joint is more involved because the mating surfaces are semi-circular and not flat. Consider a rivet subjected to a force F that is supported by the plate of the joint, as shown in Fig. 3.16. In this case, the bearing stresses are distributed around the semi-circular contact region. The bearing stresses σ_B vary from zero at point A, which lies on the horizontal axis, to a maximum at point B that is directly under the vertical centerline of the assembly.

The equilibrium equation $\sum F_y = 0$ and the symmetry of σ_B relative to θ enable us to write:

$$F = 2\int_0^{\pi/2} (\sigma_B)(t)\cos\theta(r d\theta) \qquad (3.21)$$

where t is the thickness of the plate and r is the radius of the rivet or bolt.

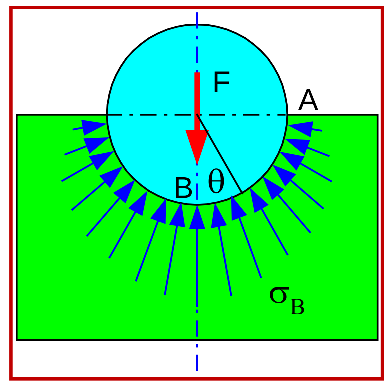


Fig. 3.16 Distribution of bearing stresses σ_B around the semi-circular contact region.

Next, we assume that the bearing stresses follow a cosine distribution over the contact region and write:

$$\sigma_B = (\sigma_B)_{Max} \cos\theta \qquad (3.22)$$

Substituting Eq. (3.22) into Eq. (3.21) and using the trigonometric identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ yields:

$$F = 2t\,r(\sigma_B)_{Max}\int_0^{\pi/2}\cos^2\theta\, d\theta = 2t\,r(\sigma_B)_{Max}\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta \qquad (a)$$

Integrating Eq. (a) and solving for $(\sigma_B)_{Max}$ gives:

$$(\sigma_B)_{Max} = \frac{2F}{\pi\, t\, r} = \frac{4F}{\pi\, t\, d} \qquad (3.23)$$

where d is the diameter of the rivet or bolt.

On the other hand, if the distribution of σ_B about the semi-circular bearing area is assumed to be uniform, we rewrite Eq. (3.21) as:

$$F = 2t\,r\sigma_B\int_0^{\pi/2}\cos\theta\, d\theta \qquad (3.24)$$

Integrating Eq. (3.24) and solving for (σ_B) gives:

$$\left(\sigma_B\right)_{Ave} = \frac{F}{2\,t\,r} = \frac{F}{t\,d} = \frac{F}{A_P} \qquad (3.25)$$

where A_P is the projected area of the semicircular region onto its horizontal axis.

Equation (3.25) is often used to estimate σ_B, because it follows the classic formula for stresses — F/A; however, Eq. (3.23) provides a more accurate estimate of the maximum bearing stress.

EXAMPLE 3.15

The lap joint with a single shear plane, shown in Fig. E3.15, is 1,000 mm wide with four bolts spaced on 250 mm centers. The two plates are each 20 mm thick and the bolts are 18 mm in diameter. Determine the bearing stress on the bolt holes if the applied force q per unit width is uniformly distributed over the width of the joint and equal to 90 N/mm.

Solution:

The force acting on a single rivet is determined from Eq. (3.11) as:

$$F = (qw)/(N) = (90)(1{,}000)/(4) = 22.5 \text{ kN} \qquad (a)$$

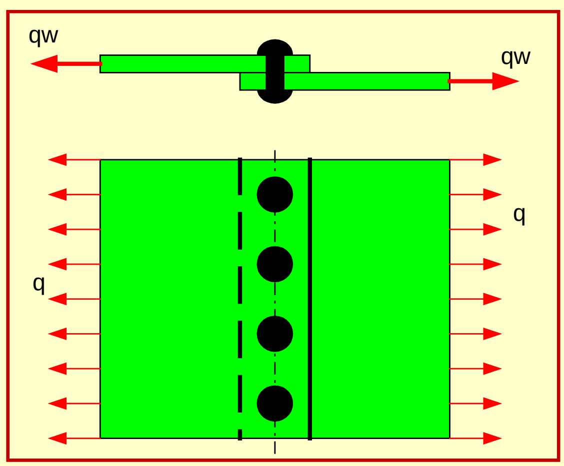


Fig. E3.15

Assuming the bearing stresses are distributed around the periphery of the bolt hole as a cosine function, we determine σ_B using Eq. (3.23).

$$\left(\sigma_B\right)_{Max} = \frac{4F}{\pi\,t\,d} = \frac{4(22.5\times10^3)}{\pi\,(20)(18)} = 79.58 \text{ MPa} \qquad (b)$$

Assuming the bearing stresses are uniformly distributed around the periphery of the bolt hole, we determine σ_B using Eq. (3.25).

$$(\sigma_B)_{Ave} = \frac{F}{t\,d} = \frac{22.5 \times 10^3}{(20)(18)} = 62.50 \text{ MPa} \qquad \text{(c)}$$

The result obtained using Eq. (3.23) is larger by a factor of $4/\pi$ or (1.273) than the result obtained with Eq. (3.25)

EXAMPLE 3.16

The butt joint with two straps, shown in Fig. E3.16, is 1,200 mm wide with four bolts spaced on 300 mm centers. The four plates are each 30 mm thick and the bolts are 35 mm in diameter. Determine the bearing stresses on the bolt holes, if the applied force per unit width q is uniformly distributed over the width of the joint and equal to 600 N/mm.

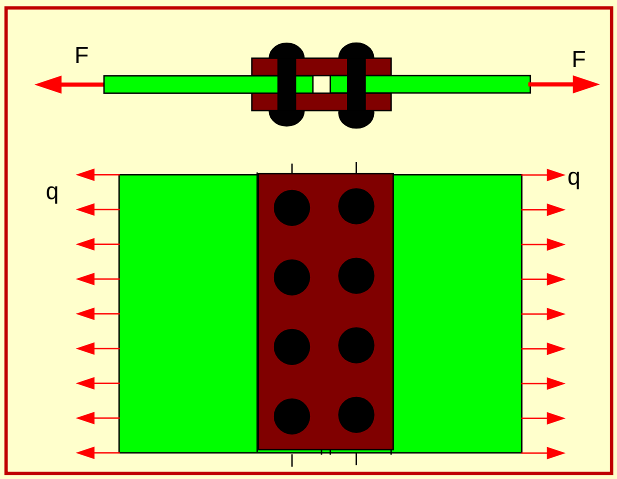


Fig. E3.16

Solution:

The force acting on a single rivet is determined from Eq. (3.11) as:

$$F = (qw)/(N) = (600)(1{,}200)/(4) = 180.0 \text{ kN} \qquad \text{(a)}$$

Assuming the bearing stresses are distributed around the periphery of the bolt hole as a cosine function, we determine σ_B using Eq. (3.23).

$$(\sigma_B)_{Max} = \frac{4F}{\pi\, t\, d} = \frac{4(180 \times 10^3)}{\pi\,(30)(35)} = 218.3 \text{ MPa} \qquad \text{(b)}$$

Assuming the bearing stresses are uniformly distributed around the periphery of the bolt hole, we determine σ_B using Eq. (3.25).

$$(\sigma_B)_{Ave} = \frac{F}{t\,d} = \frac{180 \times 10^3}{(30)(35)} = 171.4 \text{ MPa} \qquad \text{(c)}$$

The result obtained using Eq. (3.23) is larger by a factor of $4/\pi$ or (1.273) than the result obtained with Eq. (3.25)

3.5 THE TENSILE TEST

We determine several material properties, as shown in Appendix B, by conducting standardized tensile tests[4]. For example, the yield and ultimate tensile strength are determined by standardized testing. To begin, we machine **tensile specimens** from the materials under investigation. The standard size of the specimen is shown in Fig. 3.17, although in some tests smaller specimens are employed.

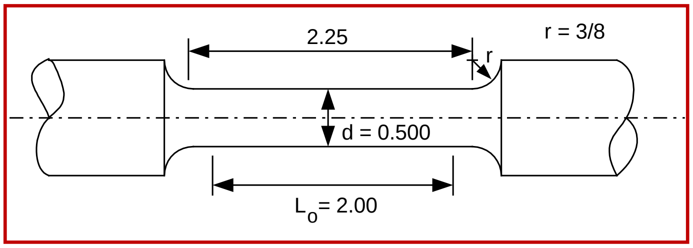


Fig. 3.17 Tensile test specimen (dimensions in inches).

The tensile specimen is mounted in a **universal testing machine** similar to the one illustrated in Appendix D-1. The testing machine may be a mechanical type with one head fixed and the other head is driven by screws, or it may be a hydraulic machine where the movable head is driven by a hydraulic cylinder. In either instance, the testing machine applies a load F along the axis of the tensile specimen. The load is applied slowly, until the specimen yields and/or fails by fracturing.

During the tension test, we measure the applied load F and the deformation δ over a gage length L_o of the specimen. A **load cell**, in the loading train shown in Fig. 3.18, measures the applied load F and an **extensometer** mounted on the specimen, shown in Fig. 3.19, measures the deformation δ. The electrical signals from the load cell and the extensometer are recorded together on a x-y chart to provide a load-deformation curve that is proportional to the stress-strain curve. The load-deflection curves recorded during the tension tests are converted into stress-strain curves, which characterize the tensile behavior of the metallic material by utilizing:

$$\varepsilon = \delta/L_o \qquad (3.3)$$

$$\sigma = F/A_o \qquad (3.4)$$

where L_o and A_o are the initial gage length and cross sectional area of the tensile specimen.

[4] The American Society for Testing and Materials (ASTM) publishes standards that define the test specimen and procedures for measuring the material properties described in this chapter. The standard for tension testing of metallic materials is E8 – 99, (see the *Annual Book of ASTM Standards* volume 03.01).

This measurement gives the **engineering stress** and the **engineering strain,** which differ from the **true stress** and the **true strain**. The distinction between the "engineering" and "true" quantities is described in Section 3.7

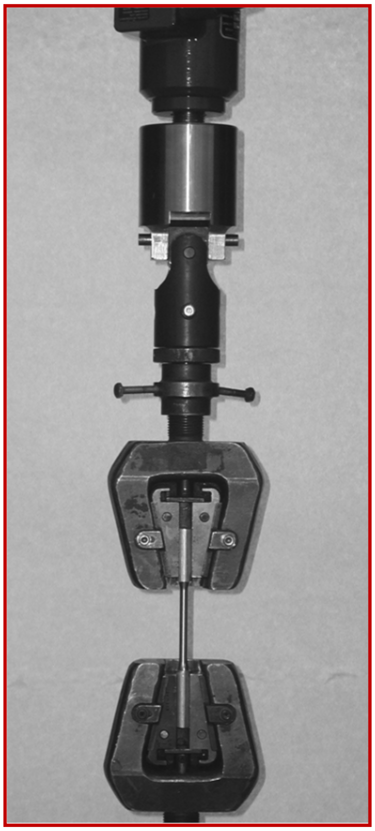


Fig. 3.18 Tensile specimen, wedge-grips, flex-joints and load cell in a testing machine.

Fig. 3.19 A tension specimen with an extensometer for measuring the deformation δ.

The stress-strain curve for brittle materials is nearly linear until failure, as indicated in Fig. 3.20 Brittle materials do not exhibit significant plastic deformation before fracture. The stress σ_f producing the failure of the brittle specimen is determined from the results of the tensile test. It is this value of the stress that defines the material property known as the ultimate tensile strength S_u for a brittle material.

$$S_u = \sigma_f \tag{3.26}$$

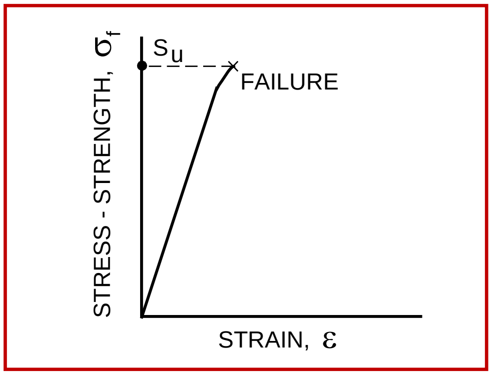


Fig. 3.20 Stress-strain curve for a brittle metallic material.

A photograph of a tension specimen exhibiting brittle failure is depicted in Fig. 3.21. Note the absence of any necking or extensive plastic deformation in the specimen. Brittle failure is dangerous, because it is sudden and catastrophic. This type of fracture initiates without warning and the cracks propagate across the specimen (structure) in microseconds. Of course, in the selection of materials for our designs, we avoid the use of brittle materials in structures to preclude the possibility of a catastrophic failure. Brittle materials such as gray cast iron are sometimes used because of its excellent casting properties and for its ability to damp vibrations; however, when using brittle materials, we are careful to maintain a state of compressive stress in the structure and to avoid tensile stresses.

Fig. 3.21 Failure of a brittle material occurs suddenly with little plastic deformation.

Structures are designed with ductile materials, which yield and undergo extensive plastic deformation prior to rupture. A typical stress-strain curve for a ductile material (mild steel) is presented in Fig. 3.22.

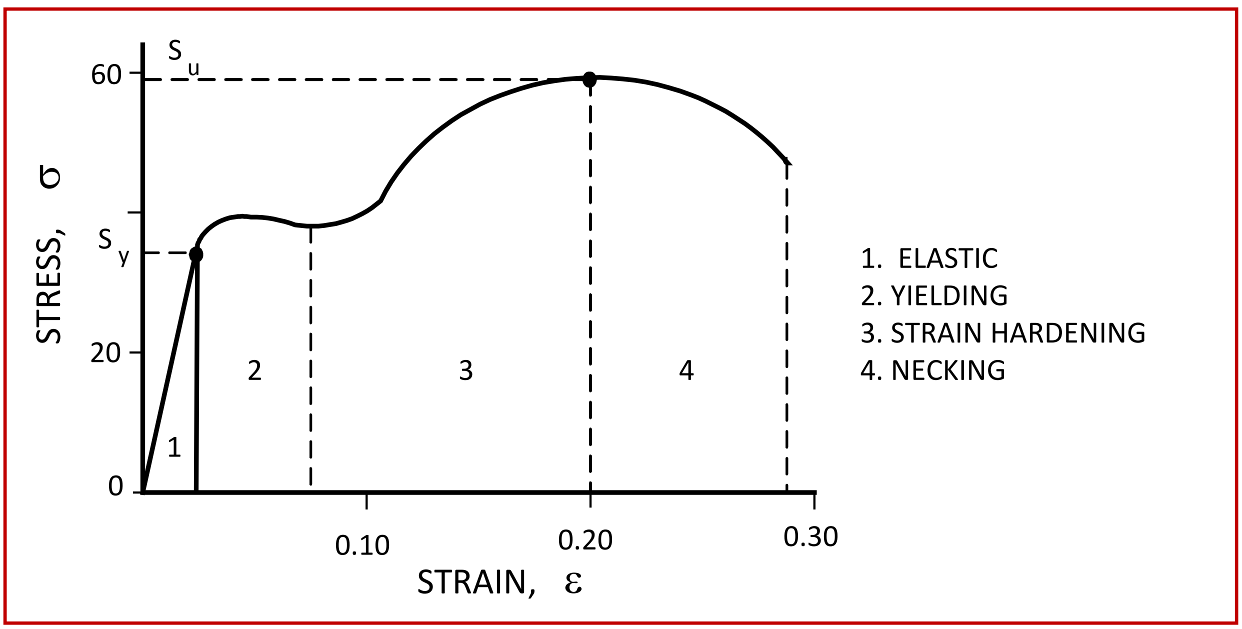


Fig. 3.22 Stress-strain diagram for low-carbon steel.

An inspection of this stress-strain diagram shows that there are four different deformation regions, each corresponding to a different behavior of ductile steel. The first is called the **elastic region,** where the material responds in a linear manner. In this region, Hooke's law applies and we write:

$$\sigma = E\,\varepsilon \tag{3.5}$$

Hooke's law is a mathematical model of material behavior, but it is valid only in the elastic region. The elastic region extends until the low carbon steel (or some other ductile material) begins to yield. When the material yields, the linear response of the material ceases and Eq. (3.5) is no longer valid.

Region (2) on the stress-strain diagram depicts the initiation of yielding, where the stress in the specimen exceeds the **elastic limit**. Slip planes develop in the specimen, and deformation by slip along these planes occurs with small or negligible increases in the stress. The stress-strain relationship is non-linear in this region. In some ductile materials, a small decrease in the applied stress occurs as the tensile specimen yields and continues to elongate by slip.

Region (3) describes a material behavior known as **strain hardening**. The easy slip that occurred during the initial yielding phase becomes more difficult to induce. As a consequence, higher stresses are required to continue deforming the tensile specimen. The stresses increase with increases in the strain; however, the relationship is not linear and Eq. (3.5) is not valid in Region (3).

In Region (4), the tensile specimen undergoes a dramatic change in appearance: it begins to **neck**. The deformation becomes localized to a small region along the length of the bar. During this phase of the deformation, the specimen resembles an hourglass. The neck decreases in diameter with increasing deformation until the specimen fails by rupturing. The axial deformation, which occurs as the tensile specimen necks, requires no increase in the applied loads. Indeed, the load usually decreases significantly during the necking phase of the deformation processes. The appearance of a ruptured tension specimen fabricated from low carbon steel is illustrated in Fig. 3.23. Regions (2), (3) and (4) are combined and called the **plastic regime** for a ductile material.

Fig. 3.23 Ductile failure of a tensile specimen. Note the cup and cone fracture region

3.6 MATERIAL PROPERTIES

The tensile test provides several material properties that are important in the analysis of engineering components and structures. These properties include two measures of strength, two measures of ductility and two elastic constants. Let's first discuss strength.

3.6.1 Measures of Strength

The two measures of strength determined in a tensile test of a ductile material are the yield strength and the ultimate tensile strength. The yield strength, as the name implies, is the stress required to induce yielding:

$$S_y = \sigma_y \tag{3.27}$$

To establish the yield stress σ_y, we examine the stress-strain diagram and attempt to identify the stress when yielding initiated. For some ductile materials with stress-strain diagrams similar to that shown in Fig. 3.22, the precise identification of σ_y is clear. However, the yield behavior of other materials is much less well defined. For example, suppose a material exhibits the stress-strain behavior, as indicated in Fig. 3.24a. Where is the yield point?

It is evident in Fig. 3.24a that the stress-strain curve is non-linear, but we might differ in defining the point where slip and yield initiated. To eliminate the ambiguity in the definition of the yield point from a stress-strain diagram, the offset method is employed. A line parallel to the linear portion of the σ-ε curve in the elastic region is constructed. This line is offset along the strain axis by 0.2% or $\varepsilon = 0.002$. The intersection of the offset line with the σ-ε curve defines the yield point, as illustrated in Fig. 3.24b. With the **0.2% yield stress** σ_y defined, we use Eq. (3.27) to establish the yield strength S_y for the material.

The ultimate tensile strength is established from the maximum stress on the specimen that occurs at the onset of necking. We have defined this point on Fig. 3.24b, and have established the ultimate stress σ_u. The ultimate tensile strength S_u for a ductile material is given by:

$$S_u = \sigma_u \tag{3.28}$$

The stress at failure, σ_f for a ductile material, shown in Fig. 3.24b, is only of academic interest. If the structure or machine component is properly designed, the deformation state will probably be limited to the elastic region. In some instances, plastic deformations are tolerated in design, but these exceptions are limited to yielding in small local regions under conditions of constant (static) loading. Global yielding is not tolerated, and yielding under the action of cyclic stresses almost always leads to failure. For this reason, the stresses imposed on a structure are usually less than σ_y and certainly less than σ_u.

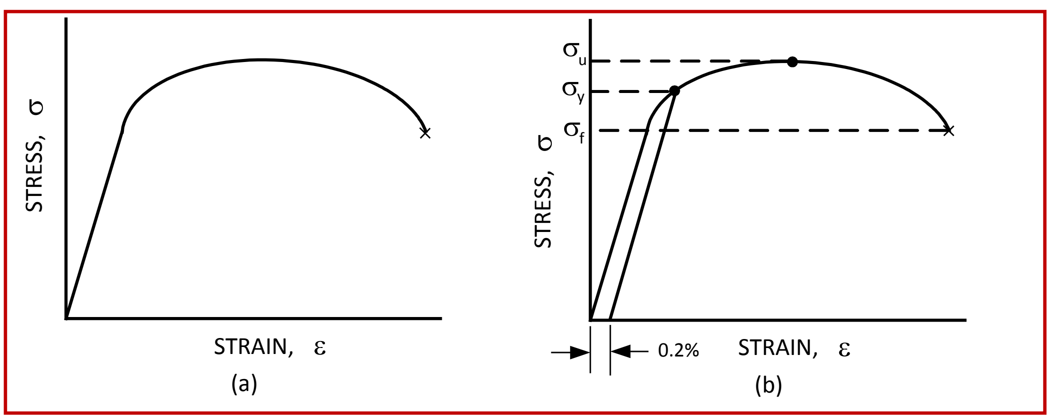


Fig. 3.24 The offset method for determining the stress σ_y for yielding.

3.6.2 Measures of Ductility

There are two common measures of ductility. The first is the **percent elongation** given by:

$$\% \, e = \frac{L_f - L_0}{L_0} \times 100 \qquad (3.29)$$

where L_o is the gage length of the tensile specimen — usually 2.0 in. or 50 mm.
L_f is the final deformed length of the specimen measured between the gage length marks.

For low carbon steels, the percent elongation is in the range from 20 to 35%; however, the percent elongation decreases with increasing carbon content in the steel and its increasing strength.

The second measure of ductility is the **percent reduction in area** that is given by:

$$\% \, A = \frac{A_0 - A_f}{A_0} \times 100 = \left[\left(\frac{d_0^2 - d_f^2}{d_0^2} \right) \right] \times 100 \qquad (3.30)$$

where A_o is the initial cross sectional area of the tensile specimen given by $\pi d_o^2/4$, and $A_f = \pi d_f^2/4$ is the final cross sectional area of one of the ruptured ends of the specimen.

The percent reduction in area for low-carbon steel is typically in the range of 60 to 70%. As was the case with percent elongation, the percent reduction in area decreases as the carbon content of steel is increased to enhance its strength.

There is a trade-off between ductility and strength for metallic alloys. To illustrate the loss in ductility with increasing strength, let's examine typical stress-strain curves for three different types of steels that are shown in Fig. 3.25. As the strength increases the strain to failure decreases with an accompanying decrease in the ductility. For low-carbon steels, the strain to failure usually exceeds 60 to 70%. This value decreases to 30 to 40% for higher carbon steels. The very high-strength steel alloys fail at strains ranging from 10 to 20%.

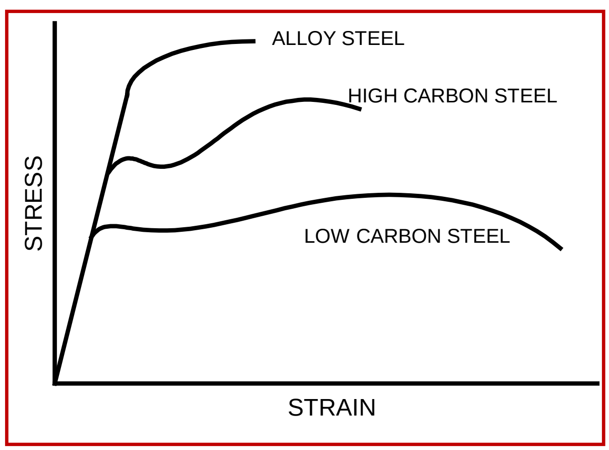


Fig. 3.25 Stress-strain curves for three different types of steels.

EXAMPLE 3.17

A steel supplier provides you with data from a recent series of tensile tests of two different steels that they sell to your employer. Your manager questions the ductility of both materials, and asks you to determine it. An examination of the supplier's data indicates:

Lower cost steel at \$22.17/100lb	Higher cost steel at \$31.82/100lb
L_f = 68 mm and d_f = 8.9 mm	L_f = 63 mm and d_f = 10.6 mm

The gage length L_o and diameter d_o for specimens from both types of steel was 50 mm and 12.5 mm respectively.

Solution:

For the measure of ductility known as the percent elongation, Eq. (3.29) gives:

$$\%e = (100)(68 - 50)/(50) = 36\% \Rightarrow \text{lower cost steel.}$$
$$\%e = (100)(63 - 50)/(50) = 26\% \Rightarrow \text{higher cost steel.} \qquad \text{(a)}$$

For the measure of ductility known as the percent reduction in area, Eq. (3.30) gives:

$$\%A = 100(d_o^2 - d_f^2)/d_o^2 = 100[(12.5)^2 - (8.9)^2]/(12.5)^2 = 49.31\% \Rightarrow \text{lower cost steel.}$$
$$\%A = 100(d_o^2 - d_f^2)/d_o^2 = 100[(12.5)^2 - (10.6)^2]/(12.5)^2 = 28.09\% \Rightarrow \text{higher cost steel.} \qquad \text{(b)}$$

When your manager asks which steel has the higher ductility, what is your response? You reply that the lower cost steel has more ductility but lower strength.

3.6.3 Elastic Constants

Modulus of Elasticity

In this discussion of elastic constants, we limit the application of these parameters to characterize deformations in the linear elastic region. In this region, the material is elastic and it recovers completely, when the load or stress is removed from the specimen. The slope of the stress-strain curve is defined as the **modulus of elasticity** or Young's modulus. The slope of the curve, illustrated in Fig. 3.26, is given by $E = \sigma/\varepsilon$.

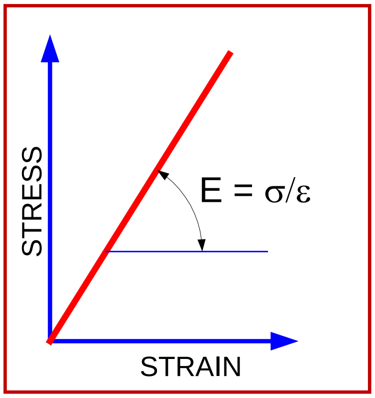


Fig. 3.26 The slope of the stress-strain curve determines the modulus of elasticity.

EXAMPLE 3.18

After conducting a tensile test with a mild steel specimen, you measure the slope $\Delta P/\Delta\delta$ of the trace on the load-deflection curve (P, δ) and find:

$$\Delta P/\Delta\delta = 9{,}450 \text{ lb}/3.16 \times 10^{-3} \text{ in.}$$

If the specimen diameter is 0.502 in. and the gage length of the extensometer is 2.00 in., determine the modulus of elasticity E.

Solution:

From Eqs. (3.5) and (3.6), we write:

$$E = \sigma/\varepsilon = (P/A)/(\delta/L) = PL/\delta A = \Delta PL/\Delta\delta A \qquad \text{(a)}$$

$$E = [(9{,}450 \text{ lb})(2 \text{ in.})]/[(3.16 \times 10^{-3} \text{ in.})(\pi\,(0.251)^2 \text{ in.}^2)] = 30.22 \times 10^6 \text{ psi} \qquad \text{(b)}$$

Poisson's Ratio

Another elastic constant, **Poisson's ratio**, can be measured in the tensile test; however, electrical resistance strain gages are attached to the specimen to measure the strain in both the axial and the transverse directions. Poisson's ratio is defined as:

$$\nu = -\,\varepsilon_t/\varepsilon_a \qquad (3.31)$$

where ε_t is the strain in the transverse direction and ε_a is the strain in the axial direction.

When a tensile force is applied to a specimen, its length increases and its diameter contracts. This contraction is usually 1/10 to 1/2 of the amount of the specimen's axial extension. Extension of the specimen, in the elastic region, produces a contraction that is usually too small to be observed during the tensile test, until necking occurs.

Poisson's ratio is a measure of the relative contraction in the specimen's diameter as it is loaded. Although too small to be observed, the Poisson effect is a very important phenomenon, because it significantly affects the analysis of structures subjected to multiaxial states of stress, where stresses are imposed in more than one direction.

To describe the Poisson contraction in more detail, consider the rectangular bar of an elastic material with dimensions L, W, and D, as shown in Fig. 3.27a. Next, apply an axial strain ε_a and write:

$$\varepsilon_a = \Delta L/L \qquad \text{(a)}$$

In the deformed state, the dimensions of the bar change to $L + \Delta L$, $W - \Delta W$, and $D - \Delta D$, as shown in Fig. 3.27b. The strains ε_t in the x and y directions (both x and y are transverse directions) are due to the Poisson effect and are equal. They are given by:

$$\varepsilon_{tx} = \Delta W/W \qquad \text{and} \qquad \varepsilon_{ty} = \Delta D/D \qquad \text{(b)}$$

If we substitute Eq. (3.26) and Eq. (a) into Eq. (b), we obtain:

$$\varepsilon_t = -\,\nu\varepsilon_a = -\,\nu\Delta L/L = \Delta W/W = \Delta D/D \qquad \text{(c)}$$

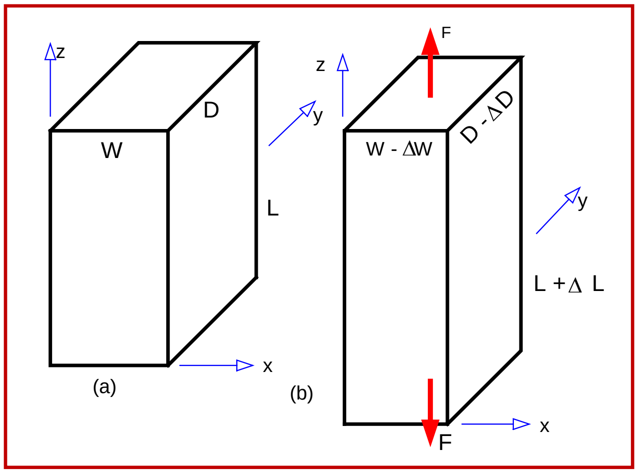


Fig. 3.27 A rectangular bar before and after imposing an axial deformation.

From Eq. (c) it is clear that the changes in the transverse dimensions of the bar due to the Poisson effect are given by:

$$\Delta W = -\nu\Delta L(W/L); \qquad \Delta D = -\nu\Delta L(D/L) \qquad \text{(d)}$$

From this elementary analysis of the deformed geometry of a rectangular bar, we note that the material property **Poisson's ratio, ν** gives the transverse deformation in terms of the axial deformation, when a structural member is subjected to a uniaxial loading. The quantities ΔW and ΔD are negative when ΔL is positive because the Poisson effect produces a contraction in the transverse (x and y) directions, when the body is extended in the axial (z) direction.

EXAMPLE 3.19

You cut a circular hole with a 80-mm diameter in a large sheet of dental dam (very thin rubber sheet). If the sheet, originally 1,100 mm long by 700 mm wide, is stretched until it is 1,250 mm long, determine the new width of the sheet and the dimensions of the deformed hole. Assume that the rubber is perfectly elastic with a Poisson's ratio, ν = ½.

Solution:

Recall Eq. (3.31) and write:

$$\varepsilon_t = -\nu\varepsilon_a = -\nu\,\Delta L/L_o = -(1/2)(1{,}250 - 1{,}100)/(1{,}100) = -0.06818 \qquad \text{(a)}$$

$$\varepsilon_a = \Delta L/L_o = (1{,}250 - 1{,}100)/(1{,}100) = 0.1364 \qquad \text{(b)}$$

Because these strains are imposed over the entire sheet, determine the deformed width W_{NEW} by:

$$W_{NEW} = W_o + \Delta W = W_o + \varepsilon_t W_o = (1 + \varepsilon_t)W_o = (1 - 0.06818)(700) = 652.3 \text{ mm} \qquad \text{(c)}$$

The dimensions of the deformed hole are given by:

$$d_a = d_o + \Delta d_a = d_o + \varepsilon_a d_o = (1 + \varepsilon_a)d_o = (1 + 0.1364)80 = 90.91 \text{ mm}$$

$$d_t = d_o + \Delta d_t = d_o + \varepsilon_t d_o = (1 + \varepsilon_t)d_o = (1 - 0.06818)80 = 74.55 \text{ mm} \qquad \text{(d)}$$

where d_a and d_t are the axial and transverse diameters of the hole after deformation.

EXAMPLE 3.20

Determine the change in volume of a rectangular bar subjected to an axial strain of $\varepsilon_a = 2.1 \times 10^{-3}$. The dimensions of the bar before deformation were L = 4W = 3D = 24 in. The material from which the bar is fabricated has a Poisson's ratio of $\nu = 1/3$.

Solution:

The original volume, $\mathbf{V}$ of the bar is given by:

$$\mathbf{V} = L \times W \times D = L^3/12 = (24)^3/12 = 1{,}152 \text{ in}^3 \qquad \text{(a)}$$

From Eq. (3.31), we determine the new dimensions of the bar as:

$$W_{NEW} = (1 - \nu\varepsilon_a)W; \quad D_{NEW} = (1 - \nu\varepsilon_a)D; \quad L_{NEW} = (1 + \varepsilon_a)L \qquad \text{(b)}$$

After substituting given information, we can rewrite Eq. (b) as:

$$W_{NEW} = (1 - \varepsilon_a/3)L/4; \quad D_{NEW} = (1 - \varepsilon_a/3)L/3; \quad L_{NEW} = (1 + \varepsilon_a)L \qquad \text{(c)}$$

The new volume is given by:

$$\mathbf{V}_{NEW} = L_{NEW} \times W_{NEW} \times D_{NEW} = (1 + \varepsilon_a)\,[1 - \varepsilon_a/3]^2\,(L^3/12) \qquad \text{(d)}$$

From Eqs. (a) and (d), it is evident that:

$$\Delta\mathbf{V} = \{(1 + \varepsilon_a)\,[1 - \varepsilon_a/3]^2 - 1\}\,(L^3/12) \qquad \text{(e)}$$

Substituting $\varepsilon = 2.1 \times 10^{-3}$ into Eq. (e) yields:

$$\Delta\mathbf{V} = [(1 + 2.1 \times 10^{-3})\,(1 - 0.7 \times 10^{-3})^2 - 1]\,\mathbf{V} \qquad \text{(f)}$$

Performing the calculation gives:

$$\Delta\mathbf{V} = 6.976 \times 10^{-4}\,\mathbf{V} = 6.976 \times 10^{-4}\,(1{,}152) = 0.8036 \text{ in}^3 \qquad \text{(g)}$$

The percentage change in the volume is given by:

$$\Delta\mathbf{V}/\mathbf{V} = [(1 + \varepsilon_a)\,(1 - \nu\varepsilon_a)^2 - 1] = 6.976 \times 10^{-4} \text{ or } 0.06976\% \qquad \text{(h)}$$

From these results it is evident that the change in the volume is extremely small for strains in the elastic region.

Shear Modulus

Another elastic constant, which is often used in the analysis of structures subjected to shear stress, is the **shear modulus**, G. The shear modulus relates the shear stress to the shear strain by:

$$\tau = G\,\gamma \qquad (3.32)$$

where τ is the shear stress and γ is the shear strain.

The shear strain γ is defined as the change in angle between two perpendicular lines, when a body is deformed. For instance, suppose we have a Cartesian coordinate system scribed on a body with the x and y axes serving as the two perpendicular scribe lines. When the body deforms under the action of a shear stress τ, these two lines rotate, and the angle at their intersection changes from 90° to some other angle 90° $\pm \gamma$, as illustrated in Fig. 3.28.

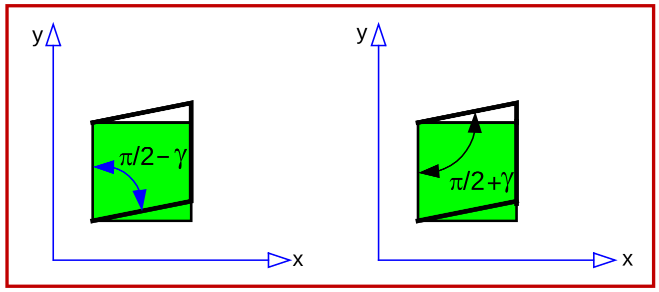


Fig. 3.28 Shear strain is the change in a right angle when a body deforms.

The shear modulus G is related to the modulus of elasticity and Poisson's ratio by:

$$G = \frac{E}{2(1 + \nu)} \tag{3.33}$$

EXAMPLE 3.21

Determine the shear modulus G for:

(a) Steel with the modulus of elasticity $E = 30 \times 10^6$ psi and Poisson's ratio $\nu = 0.30$.
(b) An aluminum alloy with E = 73 GPa and Poisson's ratio $\nu = 0.33$.

Solution:

From Eq. (3.33) we write for steel the following relation:

$$G = (30 \times 10^6) / [2(1 + 0.30)] = 11.54 \times 10^6 \text{ psi for steel.}$$

For the aluminum alloy, the shear modulus is given by:

$$G = (73 \times 10^9) / [2(1 + 0.33)] = 27.44 \text{ GPa for an aluminum alloy.}$$

3.7 TRUE STRESS AND TRUE STRAIN

In the standard tensile test, the (engineering) stress is defined as:

$$\sigma = P/A_o \tag{3.4}$$

where A_o is the cross sectional area of the tensile specimen.

When a tensile specimen is loaded, the diameter of the bar decreases due to Poisson contraction. Clearly, the cross sectional area of the specimen is changing during the standard tensile test. How do we handle this situation?

There are two approaches. First, with the engineering approach, we ignore the changes in the cross sectional area and compute the stress (engineering) using Eq. (3.4). We also determine the strain (engineering) by:

$$\varepsilon = \delta/L_o \tag{3.3}$$

where L_o is the initial gage length.

Obviously the gage length of the tensile specimen is increasing as the tensile bar is loaded. However, when the strain is treated as an "engineering" quantity, ε is computed using Eq. (3.3). The errors resulting by neglecting the changes in the area A and the gage length L are small, when the analysis is limited to stresses and strains in the elastic region

The second approach is to accommodate both the changing cross sectional area A and the gage length L in the analysis of the results from a tensile test. With this approach the "true" stress σ_T is given by:

$$\sigma_T = F/A_F \tag{3.34}$$

where A_F is the cross sectional area of the tensile specimen corresponding to the applied force F. The external force F and the internal force P were interchanged in Eq. (3.34), because P = F in a tensile specimen.

The true strain (sometimes referred to as "natural" or "logarithmic" strain) is also defined by considering an incremental change in length dL over a gage length L, which increases with the applied load. With this definition, the true strain is given by:

$$d\varepsilon_T = dL/L \tag{3.35}$$

Integrate Eq. (3.35), as the gage length increases from L_o to L_F, to obtain:

$$\varepsilon_T = \int_{L_0}^{L_F} \frac{dL}{L} = \ln \frac{L_F}{L_0} \tag{3.36}$$

The engineering strain ε is related to the true strain by:

$$\varepsilon_T = \ln (L_F/L_o) = \ln [(L_o + \Delta L)/L_o] = \ln (1 + \varepsilon) \tag{3.37}$$

If the results of a tensile test are interpreted using the definitions of true stress and true strain, the stress-strain curve in the plastic regime is changed markedly. A typical true stress-strain diagram is illustrated in Fig. 3.29. Inspection of this diagram shows that the true stress-strain curve differs dramatically from the engineering stress-strain curve. For the elastic region where the strains are very small, the two curves are nearly identical. However, for larger strains significant differences are evident. The true stress is larger than the engineering stress in the plastic region and the true strain is smaller than the engineering strain for a given load after yielding occurs. The true stress does not decrease with the onset of necking because the area at the neck decreases more rapidly than the load. The engineering stress, which is based on the initial area A_0 of the specimen, decreases because the load decreases as the neck develops.

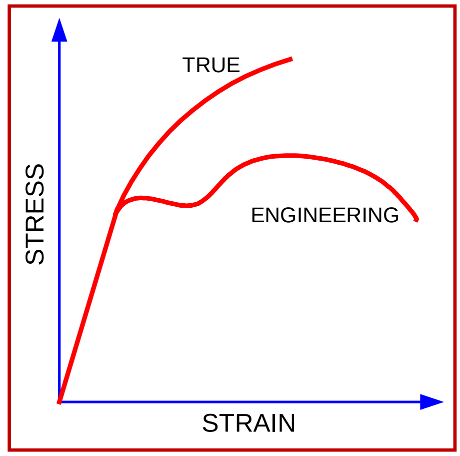


Fig. 3.29 Illustration of a true-stress—true-strain diagram.

EXAMPLE 3.22

The rupture load on a standard tensile specimen with an initial diameter d_o = 0.500 in. and initial gage length L_o = 2.00 in. was 17,500 lb. If the diameter d_F at the neck of the ruptured specimen was 0.392 in. and the gage length after failure was L_F = 2.675 in., determine:

a. The true stress at failure.
b. The true strain at failure.
c. The engineering stress at failure.
d. The engineering strain at failure.

Solution:

The true stress at failure is given by Eq. (3.34) as:

$$\sigma_T = F/A_F = 4F/\pi d_F^2 = [(4)(17{,}500)]/[(\pi)(0.392)^2] = 145{,}000 \text{ psi} \qquad \text{(a)}$$

The true strain at failure is given by Eq. (3.37) as:

$$\varepsilon_T = \ln (L_F/L_o) = \ln (2.675/2.000) = 0.2908 = 29.08\% \qquad \text{(b)}$$

The engineering stress at failure is given by Eq. (3.4) as:

$$\sigma = F/A_o = 4F/\pi d_o^2 = [(4)(17{,}500)]/[(\pi)(0.500)^2] = 89{,}130 \text{ psi} \qquad \text{(c)}$$

The engineering strain at failure is given by Eq. (3.3) as:

$$\varepsilon = (L_F - L_o)/L_o = (2.675 - 2.000)/(2.000) = 0.3375 = 33.75\% \qquad \text{(d)}$$

3.8 SUMMARY

In this chapter, you were introduced to the process of determining stress, strain and deformation in structural elements. Uniaxial members subjected to axial forces were considered first and methods of determining stress, strain and axial deformation were derived. The important relations derived are listed below:

$$\delta = L_f - L_o \quad (3.1)$$

$$\delta = (PL)/(AE) \quad (3.2)$$

$$\varepsilon = \delta/L \quad (3.3)$$

$$\sigma = P/A \quad (3.4)$$

$$\sigma = E\varepsilon \quad (3.5)$$
$$\varepsilon = P/(AE) \quad (3.6)$$

$$P = \int \sigma \, dA \quad (3.7)$$

$$P = \sigma \int dA = \sigma A \quad (3.8)$$

A second type of stress exists—a **shear stress**. As the name implies, the shear stress lies in the plane of the area on a structural member exposed by the section cut. The shear stress τ is given by:

$$\tau = V/A \quad (3.9)$$

Riveted and bolted joints were introduced and methods for solving for the shear stresses developed in the rivets and bolts.

$$\tau = V/A = F/A \quad (3.10)$$

The normal stress σ along the riveted line in a riveted joint is given by:

$$\sigma = (qw)/[t(w - Nd)] \quad (3.13)$$

The shear stress that produces pullout of the rivets from the edge of a riveted joint is given by:

$$\tau = V/A_{Shear} = F/(2A_{Shear}) = (qw)/[2N\,(t\,s)] \quad (3.14)$$

The shear stress τ in the adhesive of a lap joint fabricated with adhesive bonding is given by:

$$\tau = qw/A_A = qw/ws = q/s \quad (3.15)$$

The normal stress σ on the sheets forming an adhesively bonded lap joint is given by:

$$\sigma = qw/wt = q/t \tag{3.16}$$

The shear stress τ in the adhesive of a joint with a double lap design is given by:

$$\tau = qw/A_A = qw/2ws = q/2s \tag{3.17}$$

The maximum normal stress σ in the sheets that are bonded together is given by:

$$\sigma = qw/wt = q/t \tag{3.18}$$

The normal stress σ on the material used in making the overlaps is given by:

$$\sigma = qw/2wt = q/2t \tag{3.19}$$

The bearing stress produced over an area A_B by a force F is given by:

$$\sigma_B = F/A_B \tag{3.20}$$

The maximum and average bearing stresses on a bolt or shaft are given by:

$$(\sigma_B)_{Max} = \frac{2F}{\pi\, t\, r} = \frac{4F}{\pi\, t\, d} \tag{3.23}$$

$$(\sigma_B)_{Ave} = \frac{F}{2\, t\, r} = \frac{F}{t\, d} = \frac{F}{A_p} \tag{3.25}$$

Structures and machine components fail because of fracturing, yielding or excessive elastic deformation. In predicting either fracture or yielding, we compare the stress imposed on the structure with the strength of the material from which it is fabricated. In some analyses, we employ a handbook to provide the strength of the materials, and in more critical designs we conduct standardized tensile tests to establish the strength of the specific materials to be employed in construction.

The standard ASTM tensile test is introduced in Section 3.5. Using a standard tensile specimen and standard test procedures a force-deflection curve is generated. We convert this force-deflection data to a stress-strain curve, and then interpret the curve to give the strength, ductility and the elastic constants. Care is exercised in the interpretation to distinguish between “brittle” and “ductile” materials because the yield strength cannot be measured for brittle materials. For ductile materials, procedures are described for measuring both the yield strength and the ultimate tensile strength. The yield and ultimate tensile strengths are given by:

$$S_y = \sigma_y \tag{3.27}$$

$$S_u = \sigma_u \tag{3.28}$$

Measures of ductility are given by the percent elongation and percent reduction in area:

$$\%\ e = \frac{L_f - L_0}{L_0} \times 100 \qquad (3.29)$$

$$\%\ A = \frac{A_0 - A_f}{A_0} \times 100 = \left[\left(\frac{d_0^2 - d_f^2}{d_0^2}\right)\right] \times 100 \qquad (3.30)$$

Methods for determining both of these quantities have been given. We noted that for most types of steel the ductility decreased with increasing carbon content used for strength enhancement.

Two elastic constants, the modulus of elasticity E and Poisson's ratio ν, are measured in a tensile test. The modulus of elasticity is determined from the slope of the stress-strain curve in the elastic region. Poisson's ratio is determined from the ratio of strain in both the axial and transverse directions measured during a uniaxial tension test.

$$\nu = -\,\varepsilon_t/\varepsilon_a \qquad (3.31)$$

The shear modulus, another elastic constant often used in analysis of shear stresses and torsion loading of circular shafts, is introduced.

$$\tau = G\,\gamma \qquad (3.32)$$

$$G = \frac{E}{2(1+\nu)} \qquad (3.33)$$

Clearly, the shear modulus G is not independent, because it is a function of E and ν.

The difference between engineering and true stress and strain is described. Definitions of true stress and true strain are given, and a relation between true strain and engineering strain is derived.

$$\varepsilon_T = \int_{L_0}^{L_F} \frac{dL}{L} = \ln \frac{L_F}{L_0} \qquad (3.36)$$

$$\varepsilon_T = \ln (L_F/L_o) = \ln (1 + \varepsilon) \qquad (3.37)$$

The difference between engineering and true strains is not significant if the strains are small, but the difference becomes significant for large values of strain. For this reason, we usually employ engineering stress and strain, if we are performing analyses of structures subjected to stresses and strains in the elastic region. Structures subjected to stresses beyond the yield conditions deform significantly, and we usually perform plastic analysis using the definitions of true stress and true strain.

CHAPTER 4

AXIALLY LOADED STRUCTURAL MEMBERS

4.1 INTRODUCTION

In this chapter we will consider the analysis of two types of long-thin-structural members — flexible and stiff. Both types of these structural members will be subjected to axial loading. Wire rope or cables, which are flexible, are capable of supporting only tensile loading that tends to stretch the cable. However, rods or bars, which are stiff[1], are capable of supporting both tensile and compressive loading.

As structural elements, wire rope, cable, rods and bars subjected to axial loading have a significant advantage. They are subjected to a uniform state of stress over their cross entire sectional area. Also, if they are two-force members loaded at their two ends, the stresses are uniform over their length. Thus, the entire volume of the rod or bar is subjected to the same stress, which is the optimum condition for minimum weight design.

Shear and normal stresses occurring on inclined planes are introduced by considering a uniaxial bar with section cuts that produce FBDs with an oblique plane. Equations for determining the normal and shearing stresses on these oblique planes are derived. Examples are presented to demonstrate computational techniques.

Uniaxial members that are stepped along their length are considered and methods for determining the stresses and deformation in each portion of the bar are shown. Tapered bars are also introduced where the cross section of the bar changes as a function of position along its length. Equations are derived for this type of member, enabling the determination of the stresses and axial deformation.

Methods for determining the stresses, strains and deflection of bars and rods subjected to axial tension or compression forces are relatively simple, when the bar can be isolated and the forces acting on it are determined. However, if the bar is part of a complex structure, determining the forces acting on the bar may be difficult. In these cases, scale models of the structure under consideration are constructed and then subjected to various loads to verify structural integrity. Interpretation of the results of testing these scaled structures requires the use of scaling equations based on the geometric scale factors. Equations are derived showing the scaling relations for both stress and deformation.

4.2 DESIGN ANALYSIS OF WIRE, RODS AND BARS

We have defined two different strengths (i.e. yield and ultimate tensile) in Sections 3.5 and 3.6. Because a structural member may fail by excessive deformation, we may choose to limit the stress applied to the structure so that it is less than the yield strength S_y. On the other hand, some structures can tolerate plastic deformation in one or more members without compromising the structure's function. In these cases, we can tolerate stresses exceeding the yield strength, S_y but the stresses must be less than the ultimate strength, S_u. For example, when designing a bridge, it is important that its shape remain fixed under normal service loads. In this case, the yield strength is specified, as the maximum limit for the design stress for the bridge, to prevent post-yield deformations. However, in designing a bridge to remain

[1] Rods or bars actually extend or compress under the action of axial loads, but this deformation is small and is neglected in determining the loads and stresses. However, this very small deformation is considered when determining the strain and deformation occurring in these structural members.

intact during a collision with a tugboat or during an earthquake, the ultimate strength of the structural members is the important criterion.

When designing a structural member to carry a specified load, we always size the member so that the **design stress**, σ_{Design}, is less than the strength based on either a yield or some other failure criteria. It would not be prudent to permit the design stress to equal the strength of the member. To size structural members so they are safe, a **factor of safety, SF** is employed in the analysis. The safety factor is defined as the ratio of strength divided by the design stress. This definition leads to the relations given below:

$$\mathbf{SF_u} = S_u / \sigma_{Design} \tag{4.1}$$

$$\mathbf{SF_y} = S_y / \sigma_{Design} \tag{4.2}$$

$$\sigma_{Design} = P_{Design}/A \tag{4.3}$$

where S_u and S_y are the ultimate strength and yield strength of the material, respectively.
σ_{Design} and P_{Design} are the design stress and the design load, respectively.

The value for the factor of safety depends upon the application. Ideally, you would like the factor of safety to be as large as possible. However, excessive safety factors must be balanced by practical considerations such as economics, aesthetics, functionality and ease of assembly. In designing a structure, you will be concerned with the cost and weight of its components in addition to safety. The factor of safety specified during the design process should reflect these concerns.

4.2.1 Design Analysis of Wire and Cables

In most of the previous examples, solutions were limited to the determination of the stresses developed under specified loading conditions. Determining the stress is the first step in the design process; however, it is necessary to extend the analysis to determine if these stress levels are safe or if the structural element may fail in service. Safety factors, defined in Eq. (4.1) and Eq. (4.2), are used when designing a structural element to insure that the stresses developed under the specified loads are less than the strength of the material from which the element is fabricated. Other design analyses are employed to select the materials from which the structural element is fabricated. To illustrate the design procedure, a number of examples are presented for uniaxial members subjected to axial loading. We begin with examples for design analyses involving wire and cables, which can carry tensile stresses but not compressive stresses. Later in this section we will provide examples of design analyses for bars and rods that are capable of carrying both tensile and compressive stresses.

EXAMPLE 4.1

A hoisting cable with a diameter of 3/8 in. is fabricated from many strands of an improved plow steel wire. The manufacturer of the wire certifies its breaking load as 10,600 lb. Determine the strength of the wire used in the manufacture of the cable. Also discuss the assumption pertaining to the wire rope made in the analysis and its implication on the strength of the strands of wire.

Solution:

Recall Eqs. (3.4) and (3.26), which give:

$$S_u = \sigma_u = P_{Fail} / A \tag{a}$$

The cross sectional area of the cable will be less than $A = \pi(3/8)^2/4 = 0.1104$ in^2. Substituting this value for the area A into Eq. (a) yields:

$$S_u = (10{,}600 \text{ lb})/(0.1104 \text{ in.}^2) = 95{,}970 \text{ psi} = 95.97 \text{ ksi} \qquad \text{(b)}$$

We have assumed the cross sectional area of the wire rope is equivalent to that of a solid wire with a diameter of 3/8 in. However, wire rope is made of many very small diameter wires that are twisted together to form strands. The strands are formed in a helix about a fiber core. The wire rope in this example has a designation of 6 × 19 (6 strands with 19 small wires in each strand) and its breaking load is specified as 10,600 lb. The cross section of this wire rope is illustrated in Fig. E4.1.

The assumption of a solid cross sectional area overestimates the actual cross sectional area, A by a factor of more than two. The ultimate tensile strength of the small diameter wire used to form the strands of wire for a typical cable is usually in excess of 200 ksi.

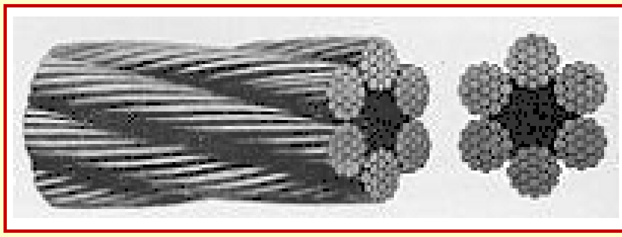


Fig. E4.1 A wire rope is fabricated from many strands of small-diameter high-strength wire.

EXAMPLE 4.2

A solid hard-drawn copper wire exhibits an ultimate tensile strength of 380 MPa. If the diameter of the wire is 4.0 mm, determine the load required to break the wire.

Solution:

Recall Eqs. (3.4) and (3.26), which give:

$$S_u = \sigma_u = P_{Fail}/A \qquad \text{(a)}$$

Rearrange this relation and substitute known quantities to give:

$$P_{Fail} = S_u A = (380 \times 10^6 \text{ N/m}^2)[\pi (4.0 \text{ mm})^2/4](10^{-6} \text{ m}^2/\text{mm}^2) = 4{,}775 \text{ N} \qquad \text{(b)}$$

An axial load in excess of 4,775 N is required to rupture the hard-drawn copper wire.

EXAMPLE 4.3

A monofilament, nylon fishing line is rated at 20-lb test. If the line is 0.42 mm in diameter, determine the strength of the nylon used to manufacture this fishing line. Comment on the effect that the filament geometry has on the strength of a polymer like nylon.

Solution:

Recall Eqs. (3.4) and (3.26), which give:

$$S_u = \sigma_u = P_{Fail}/A \qquad \text{(a)}$$

$$S_u = (20\ \text{lb})(25.4\ \text{mm/in})^2/[\pi\ (0.42\ \text{mm})^2/4] = 93{,}130\ \text{psi} \qquad \text{(b)}$$

Monofilaments of polymers are drawn from a melt into thin fibers or lines. In this process, the long molecules of the polymer are aligned with the axis of the filament, thus enhancing its strength.

EXAMPLE 4.4

A No. 14 gage (0.080 in. diameter) black annealed steel wire is listed in a materials handbook with a yield strength of 234 MPa and an ultimate tensile strength of 370 MPa. Determine the axial load that will cause the wire to yield.

Solution:

Recall Eqs. (3.4) and (3.27), which give:

$$S_y = \sigma_y = P_{Yield} / A \qquad \text{(a)}$$

$$P_{Yield} = S_y\ A \qquad \text{(b)}$$

Because we have mixed units in the problem statement, let's convert the units for the diameter of the wire to the SI system and determine the cross sectional area A in mm^2.

$$A = \pi d^2/4 = \pi\ (0.080\ \text{in.})^2\ (25.4\ \text{mm/in.})^2/(4) = 3.243\ \text{mm}^2 \qquad \text{(c)}$$

Next, substitute Eq. (c) into Eq. (b) to obtain:

$$P_{Yield} = S_y\ A = (234\ \text{MPa})[(\text{N/mm}^2)/\text{MPa}](3.243\ \text{mm}^2) = 758.9\ \text{N} \qquad \text{(d)}$$

EXAMPLE 4.5

A single #20 gage steel wire, with a solid cross section of 0.0348 in. in diameter, is intended to carry a load of 60 N with a safety factor of 4.0. Specify the steel alloy from which the wire should be manufactured to avoid failure by rupture.

Solution:

Combine Eqs. (4.1) and (4.3) to obtain an expression for S_u:

$$\mathbf{SF_u} = AS_u\ /P_{Design} \qquad \text{(a)}$$

$$S_u = (P_{Design})\mathbf{SF_u}/A = (60\ \text{N})(4.0)/[\pi\ (0.0348\ \text{in.})^2(25.4\ \text{mm/in.})^2/4] = 391.1\ \text{MPa}$$

It is necessary to refer to a materials handbook to select a suitable material for this application. Reference to Appendix B-2 indicates five steel alloys and four stainless steel alloys with ultimate tensile strengths exceeding the requirement for S_u in this example. Select alloy steel type 1212HR with an ultimate strength of 424 MPa, which is closest to the desired strength.

EXAMPLE 4.6

A cold drawn alloy steel wire, with a yield strength of 120,000 psi, is to carry a load of 725 lb while incorporating a safety factor of 3.5. You are to email the purchasing representative specifying the size of the wire to be ordered.

Solution:

To determine the size of the wire in this application, recall Eq. (4.2).

$$\mathbf{SF_y} = S_y / \sigma_{Design} = AS_y / P_{Design} \qquad \text{(a)}$$

Solve Eq. (a) for the area to obtain:

$$A = \mathbf{SF_y} (P_{Design})/S_y = \pi d^2/4 \qquad \text{(b)}$$

$$A = (3.5)(725)/(120{,}000) = 0.02115 \text{ in}^2 = \pi d^2/4 \qquad \text{(c)}$$

$$d = 0.1641 \text{ in.} \qquad \text{(d)}$$

Do you email the purchasing representative and specify cold drawn alloy steel wire with a diameter of 0.1641 in? **No!!!** If you make this mistake, you will soon learn that steel wire is available only in standard diameters. Economic laws dictate that the most cost effective way to mass-produce materials is by limiting inventories to only those sizes and geometries that satisfy most customers. In Appendix A standard wire sizes and their corresponding gage numbers are listed. Four entries from these listings are presented below:

Gage No.[2]	Diameter (in.)	Area (in^2)	Gage No.	Diameter (in.)	Area (in^2)
7	0.1770	0.02461	9	0.1483	0.01727
8	0.1620	0.02061	10	0.1350	0.01431

The response to the purchasing representative is to procure Gage No. 7 wire. It is important for you to specify a standard size in ordering structural members. Standard sizes are available with minimum delay at the lowest cost. Numerical results rarely yield dimensions for structural members that correspond to the available standard sizes. The usual practice is to increase the size determined in the analysis to the next larger standard dimension available. This approach enhances safety while minimizing cost and delivery time.

[2] There are several different standards that refer to gage numbers for wire and steel sheet. In this example, we list the American Steel and Wire standard that is commonly used for steel wire.

4.2.2 Design Analysis of Rods and Bars

All of the equations derived for wire and cable are also valid for rods and bars. If the external axial forces are tensile (tending to pull the bar apart), the internal forces and stresses are tensile and denoted with a positive sign. On the other hand, if the external axial forces are compressive (tending to smash the bar), the internal forces and stresses are compressive and denoted with a minus sign.

The relations derived for the stresses and deformations of cable assumed that the internal and external forces acted through the **centroid** of its cross sectional area. For cables or wire, with circular cross sections, the location of the centroid is obvious — it is at the center of the circle. However, for cross sections of more complex shapes, the location of the centroid is not always evident.

A **centroid** is the point coinciding with the center of gravity of a two-dimensional area. For the circular cross section, the center of the circle clearly locates the centroid. For cross sectional shapes such as ellipses, circles, squares and rectangles, the center may be located by inspection, because these shapes are all symmetric about both horizontal and vertical axes. The centroid is located at the intersection of the two axes of symmetry. However, for non-symmetric figures, such as a triangle, a portion of a circular area, a parabolic area, etc., locating the center of the gravity of the area by inspection is not possible. Methods for locating the centroid are presented in Chapter 6.

We must qualify the capability of a rod or bar to carry compressive loads. If the rod is very long and slender and the compressive force too high, the rod may buckle. **Buckling** is an unstable condition, and if the critical load is exceeded, the rod fails suddenly and catastrophically. In this chapter, we will assume that the rods or bars loaded in compression are sized to resist buckling; consequently, they fail due to excessive compressive stresses. However, the tendency for these structures to buckle cannot be ignored. Theories describing the failure of **columns** due to buckling are introduced later in Chapter 7.

The procedure for determining the normal stresses σ is the same in bars or rods as described previously for wires and cables except for the fact that the internal forces may be compressive as well as tensile. We begin by constructing a FBD to show the point of application and the directions of the internal and external forces. From the equilibrium relations, the internal forces and their sign are determined. Finally, Eq. (3.4) is employed to determine the normal stresses σ. This procedure is demonstrated in three example problems presented below.

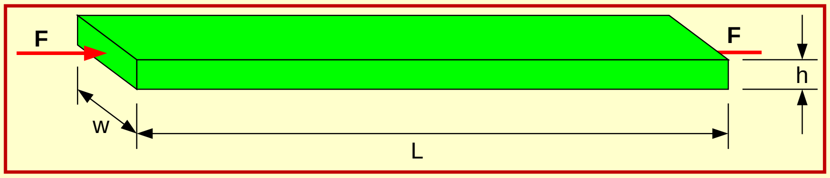
EXAMPLE 4.7

Let's reconsider Example 3.6 where we previously determined the stress in the bar shown in Fig. E3.6. The following numerical parameters define the bar and the applied load: F = 700 kN, L = 900 mm, w = 125 mm and h = 80 mm. We made a section cut near the center of the bar, constructed the free body diagram and determined the normal stress due to the 700 kN axial load as:

$$\sigma = P/A = (-\,700 \times 10^3 \text{ N})/[(125)(80) \text{ mm}^2] = -\,70.00 \text{ N/mm}^2 = -\,70.00 \text{ MPa}$$

If the bar is fabricated from 1212 HR steel, determine the safety factor based on the yielding criterion.

Fig. E3.6



Solution:

From Table B-2, the yield strength S_y is given as 193 MPa.

The safety factor is determined from Eq. (4.2) as:

$$\mathbf{SF_y} = S_y / \sigma_{Design} = (193)/(70.00) = 2.757 \qquad \text{(a)}$$

Note that the sign of the stress, a negative number, was ignored in this determination. We have assumed that the compressive and tensile strengths are the same for this material, which is true for most steels. However, this is not the case for very brittle materials such as gray cast iron.

EXAMPLE 4.8

Determine the stress in a three-foot long rod that has a diameter of 3.2 in. The bar is fabricated from mild steel with a yield strength S_y = 38,000 psi, and is subjected to an axial compressive load of 77,000 lb. Also, compute the safety factor of the bar against failure by yielding.

Solution:

From Eq. (3.4), we write:

$$\sigma = P/A = P/(\pi r^2) = (-\ 77{,}000\ \text{lb})/[\pi\ (1.6\ \text{in})^2] = -\ 9{,}574\ \text{psi} \qquad \text{(a)}$$

We determine the safety factor against yielding from Eq. (4.2).

$$\mathbf{SF_y} = S_y/\sigma_{Design} = (-\ 38{,}000)/(-\ 9{,}574) = 3.969 \qquad \text{(b)}$$

The safety factor is always a positive quantity. In this example, we compared a compressive strength of – 38,000 psi (taken as a negative quantity) with the design stress of – 9,574 psi.

EXAMPLE 4.9

In Example 3.9, we determined the shear stresses in a key that was employed to keep a gear from slipping on a shaft when transmitting power. The previous solution involved two FBDs and showed the shear stresses on the key as:

$$\tau = V/A = (50 \times 10^3)/(12)(75) = 55.56\ \text{N/mm}^2 = 55.56\ \text{MPa}$$

If the yield strength, S_y of the steel used in manufacturing the key was 315 MPa, determine the yield strength in shear and the safety factor for the key.

Solution:

The average shear stress acting on the key is 55.56 MPa. To analyze the impact of this solution on the design of a structure, it is necessary to compare the imposed stress with the shear strength of the material from which the key is fabricated. The shear strength is usually lower than the yield or ultimate tensile strength of a material. A common practice is to consider either the yield or tensile strength of a material and multiply that value by 0.5774 to estimate the shear strength.

Because the tensile yield strength, S_y of the steel used in manufacturing the key was 315 MPa, the yield strength in shear S_{ys} is estimated as:

$$S_{ys} = 0.5774\ S_y \tag{4.4}$$

$$S_{ys} = (0.5774)(315) = 181.9\ \text{MPa} \tag{a}$$

The safety factor for the key is determined from:

$$\mathbf{SF}_{ys} = S_{ys}/\tau_{Design} \tag{4.5}$$

$$\mathbf{SF}_{ys} = 181.9/55.56 = 3.274 \tag{b}$$

When interpreting the results for normal stresses, recognize the difference between tensile (+) and compressive (–) values for the results. The reason for the careful distinction is the fact that the strength of material subjected to tensile or compressive stresses is often different. However, shear strength is not sensitive to the sign of the shear stress τ. The strength of a material subjected to an imposed shear stress is not dependent on the direction of the imposed shear force. For this reason, we neglect the sign of the shear force in our analysis of shear stress on machine components.

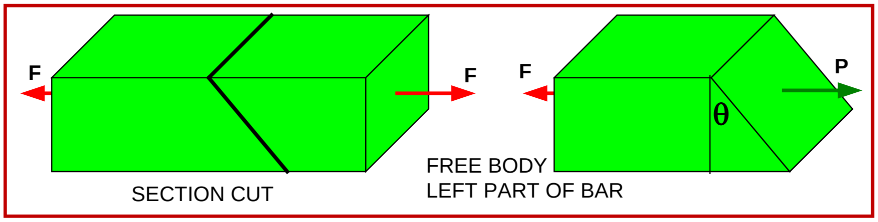
4.3 STRESSES ON OBLIQUE PLANES

In previous discussions, we have limited our choices to sections perpendicular to the axis of the bar when making section cuts. This restriction was helpful, because it simplified the state of stress observed. With these perpendicular section cuts, only normal stresses were found acting on the area exposed by the cut, because the shear stresses vanish.

We showed that shear stresses exist in Section 3.3 and computed their magnitude in Example 3.9. In this case, we restricted the section cut to one parallel to the imposed system of equal and opposite forces. With this restriction on the section cut, only shear forces were developed on the key. By restricting the section cut, we develop special cases where normal stresses exist in the absence of shear stresses and vice versa.

Let's treat the more general case with the bar cut at an arbitrary angle, as shown in Fig 4.1.

Fig. 4.1 Bar with axial loading with a section cut at an arbitrary angle.



Let's cut the free body of Fig. 4.1 with still another section cut to produce a triangular shape as shown in Fig. 4.2a. The triangular shape in Fig. 4.2b is defined with the angle θ. The left side of this triangle has an area A; therefore, the hypotenuse side of the triangle has an area $A_\theta = A/\cos\theta$. Note, the x-axis is coincident with the axis of the bar; the n axis is coincident with the outer normal to the inclined cut; the t axis is tangential to the inclined cut.

As shown in Fig. 4.2d, the internal force P has been resolved into components along the n and t axes to yield:

$$P_n = P\cos\theta \tag{a}$$

$$V_t = P \sin \theta \qquad \text{(b)}$$

To ascertain the stresses on the inclined surface, divide either P_n or V_t by the area A_θ formed with the inclined section cut. The stress σ_θ in the normal direction (n) is determined from Eqs. (3.4) and (a) as:

$$\sigma_\theta = \frac{P_n}{A_\theta} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta \qquad \text{(c)}$$

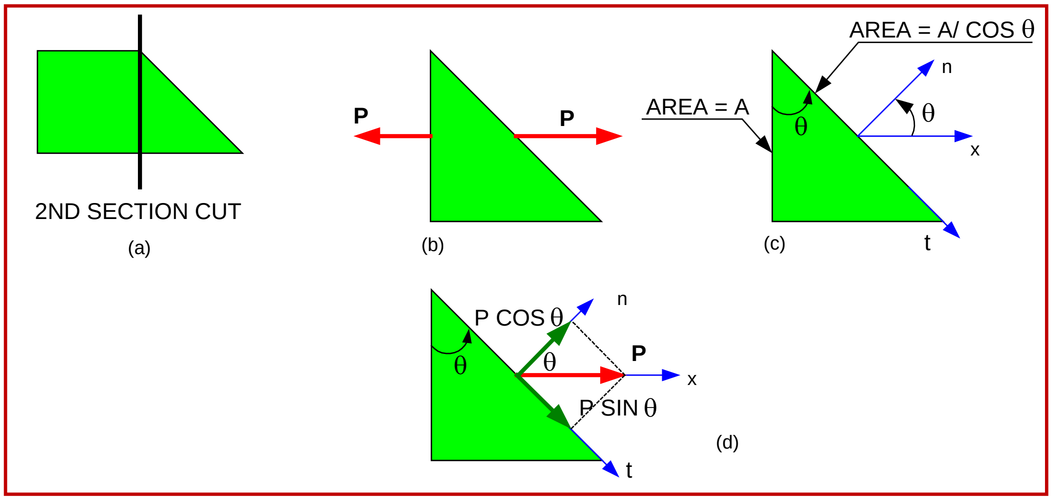


Fig. 4.2 Free body diagrams of the bar with an inclined cut:
(a) The section cut is perpendicular to the axis of the bar producing a triangle.
(b) Internal forces are acting on both faces of the triangular element.
(c) The areas of the left side and the hypotenuse of the triangle are given.
(d) The force acting on the surface of the inclined cut is resolved into components.

Substituting $P = \sigma_x A$ into Eq. (c) we obtain:

$$\sigma_\theta = (\sigma_x) \cos^2 \theta \qquad (4.6)$$

The shear stress τ_θ that acts along the face of the inclined surface is given by Eqs. (3.9) and (b) as:

$$\tau_\theta = \frac{V_t}{A_\theta} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta \qquad \text{(d)}$$

Using the trigometric identity sin θ cos θ = ½ sin 2θ, this relation is rewritten as:

$$\tau_\theta = (\sigma_x/2) \sin 2\theta \qquad (4.7)$$

Numerical results are presented in Fig. 4.3 for both σ_θ and τ_θ, as the angle of the inclined section varies from zero (a perpendicular cut) to 90° (a parallel cut). These results correspond to an axial stress $\sigma_x = 100$ units. From Fig. 4.3 it is evident that the normal stress σ is a maximum when θ = 0° and the section cut is perpendicular to the axis of the bar. For this angle of the cut, $\sigma_\theta = \sigma_x = P/A$. Thus, when employing Eq. (3.4) to compute the normal stress, we determine the maximum possible value of σ_θ. The shear stress is zero on the plane defined by θ = 0°, where the normal stress is a maximum.

Both the normal and shear stress vanish when the section cut is made at 90°. The shear stress is a maximum when the section cut is defined by $\theta = 45°$. When $\theta = 45°$, the normal and shear stresses are equal to each other $\sigma_{45} = \tau_{45} = \sigma_x/2$.

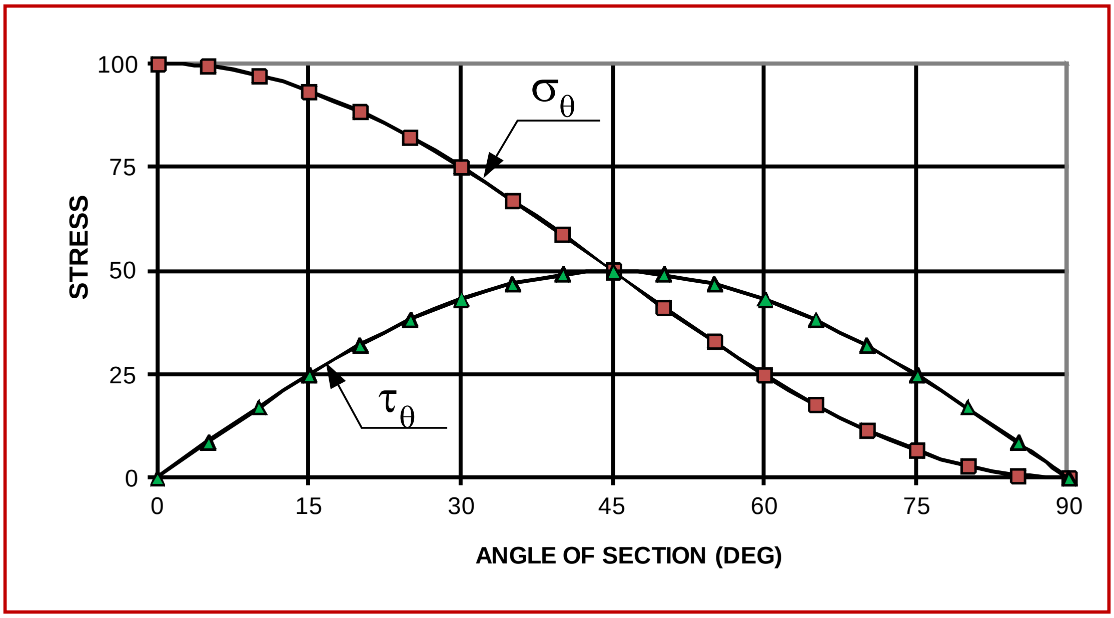


Fig. 4.3 Variation in σ_θ and τ_θ with the angle of the inclined section.

The example of the axially loaded bar, with an inclined section cut, illustrates why stresses must be treated as tensor quantities. As we vary the angle of inclination of the section cut, two parameters are changing that relate to the normal or shear stresses. First, the magnitude of the force components in the n and t direction changes with the angle θ, because forces are vector quantities. Second, the area of the inclined surface exposed by the section cut increases with θ. Both of these parameters affect the magnitude of the normal and shear stresses; therefore **stresses must be treated as tensor quantities** — not vectors.

EXAMPLE 4.10

A circular rod 24 mm in diameter and 2.2 m long is subjected to an axial load of 92 kN. Determine:

- The maximum normal stress and the plane upon which it acts.
- The maximum shear stress and the plane upon which it acts.
- Draw a FBD of the section for the maximum normal stress.
- Draw a FBD of the section for the maximum shear stress.

Solution:

The maximum normal stress occurs on a plane perpendicular to the axis of the rod where $\theta = 0°$. Equation (4.6) applies.

$$\sigma_{\theta=0} = \sigma_x \cos^2\theta = (P/A)\cos^2\theta = (92 \times 10^3)/\{[\pi\,(12)^2]\cos^2 0°\} = 203.4 \text{ MPa} \qquad \text{(a)}$$

The maximum shear stress occurs when θ = 45° as indicated by the results depicted in Fig. 4.3. From Eq. (4.7), we write:

$$\tau_{\theta=45} = (\sigma_x/2)\sin 2\theta = [(203.4)/(2)]\sin 90° = 101.7\text{ MPa} \qquad \text{(b)}$$

The FBDs for the right portion of the bar showing the maximum normal and shear stresses are shown below:

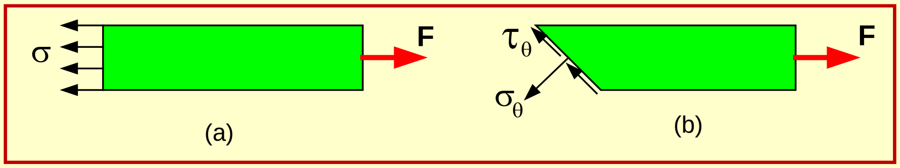


Fig. E4.10 Section cuts for (a) maximum normal stress and (b) maximum shear stress.

The FBD in Fig. E4.10a shows the normal stresses σ acting on a plane area perpendicular to the axis of the rod. The normal and shear stresses are shown in Fig. E4.10b, where they act on a surface inclined at a 45° angle to the axis of the rod.

4.4 AXIAL LOADING OF A STEPPED BAR

In some structures, bars with different cross sectional areas are employed, where the area changes abruptly at some position along the length of the bar. An illustration of a stepped bar, where the cross section undergoes an abrupt change is presented in Fig. 4.4.

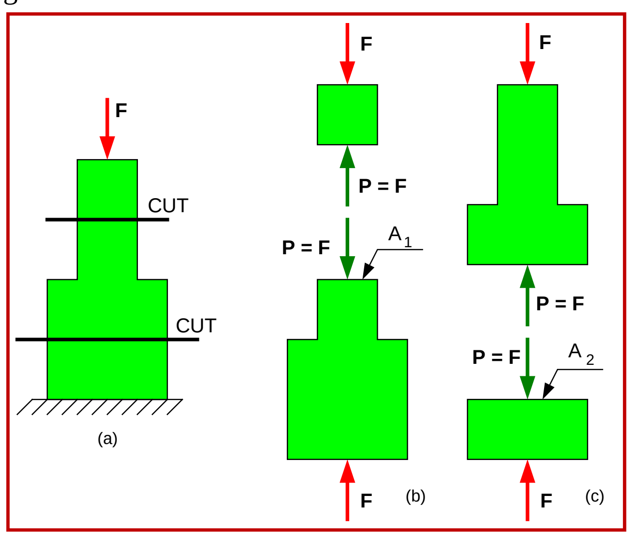


Fig. 4.4 A stepped bar with free body diagrams for each section of the bar.

The procedure for determining the normal stresses and deflection remains the same as that used for a bar with a uniform cross sectional area:

1. Draw a free body diagram of each section of the bar.
2. Use the equilibrium relations to establish the internal axial forces acting on each section.
3. Determine the stresses with σ = P/A using the area of the section of interest.
4. Solve for the deflection of each segment of the bar with δ = PL/AE, and then add the individual deflections to obtain the total deflection.

4.4.1 Normal Stresses in Stepped Bars

The normal stress varies with the cross sectional area of the bar. For example, in Fig 4.4 the normal stress in the upper section of the bar is given by $\sigma_1 = P/A_1$, and the normal stress in the lower section of the bar is $\sigma_2 = P/A_2$.

EXAMPLE 4.11

For the stepped bar shown in Fig. E4.11, determine the normal stress in each of the two sections of the bar if F_1 = 250 kN; F_2 = 125 kN; w_1 = 70 mm; b_1 = 80 mm; w_2 = 130 mm; b_2 = 80 mm; L_1 = 300 mm and L_2 = 450 mm

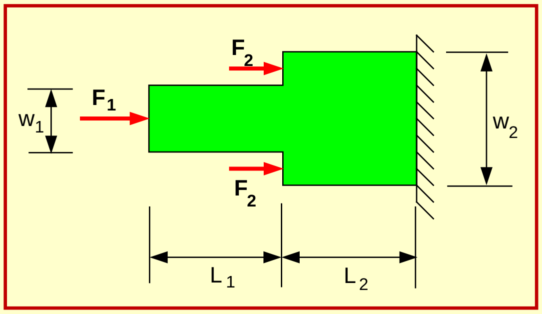


Fig. E4.11 The thickness of the bar is given by b_1 and b_2.

Solution:

Let's begin the solution by drawing free bodies of both sections of the bar. We make two section cuts A and B as illustrated in Fig. E4.11a. Then we draw free bodies associated with the portion of the bar to the left side of the section cut as shown below:

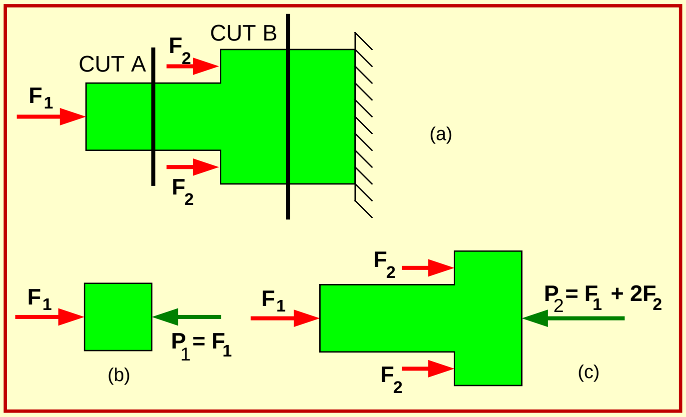


Fig. E4.11a FBDs of the two parts of the stepped bar.

From the equilibrium relations, it is clear that the internal force $P_1 = F_1$ in the smaller section of the stepped bar. Also, for the larger section of the bar, the internal force $P_2 = F_1 + 2F_2$. Finally, for the smaller section of the bar, we write:

$$\sigma_1 = P_1/A_1 = -(250 \times 10^3)/[(70)(80)] = -44.64 \text{ N/mm}^2 = -44.64 \text{ MPa} \qquad \text{(a)}$$

For the larger section of the bar, we write:

$$\sigma_2 = P_2/A_2 = -\{[250 + (2)(125)] \times 10^3\}/[(130)(80)] = -48.08 \text{ N/mm}^2 = -48.08 \text{ MPa} \qquad \text{(b)}$$

Note, the minus sign indicates the stresses in the bar are compressive.

4.4.2 Deflection of Stepped Bars

To compute the axial extension or compression of the stepped bar, we consider each uniform section of the bar separately. Equation (3.2) is valid for each section because each has a uniform cross sectional area. For a stepped bar comprised of n uniform sections, we superimpose the individual deflections δ to obtain:

$$\delta_{total} = \delta_1 + \delta_2 + \delta_3 + \ldots.. + \delta_n \qquad (4.8)$$

EXAMPLE 4.12

If the stepped bar defined in Example 4.11 is fabricated from a titanium alloy with a modulus of elasticity E = 114 GPa = 114 × 10^3 MPa, determine the total deflection of the bar.

Solution:

We recognize that the deflection of the stepped bar is determined from Eqs. (4.8) and (3.2) as:

$$\delta_{total} = \delta_1 + \delta_2 = \left[\frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E}\right] \qquad (a)$$

Because the modulus of elasticity is constant for both sections of the bar, Eq. (a) reduces to:

$$\delta_{total} = \delta_1 + \delta_2 = \frac{1}{E}\left[\frac{P_1L_1}{A_1} + \frac{P_2L_2}{A_2}\right] \qquad (b)$$

Substituting the numerical parameters for the unknown quantities in this relation yields:

$$\delta_{total} = \delta_1 + \delta_2 = \frac{1}{114\times10^3}\left[\frac{(-250\times10^3)(300)}{(70)(80)} + \frac{(-500\times10^3)(450)}{(130)(80)}\right] = -0.3073 \text{ mm}$$

Let's interpret this solution. The negative sign indicates the bar was compressed and the deformations reduced its length. The original length of the bar was $L_1 + L_2$ = 750 mm. When the total deformation of the bar is compared to this length, we find the deformation is very small—only 0.041%. This example again emphasizes that the axial deformation of metallic members is usually very small. For this reason, we neglect these deformations and use the original lengths of structural members when substituting into the equilibrium equations.

4.5 AXIAL LOADING OF A TAPERED BAR

The discussion of stress and deflection of a rod or bar has been limited to those members with a uniform cross sectional area. These uniform bars and rods are most commonly employed in building structures, because they are usually commercially available and less expensive to manufacture. However, in some instances where weight is important, it is more efficient to use members that are not uniform. In these cases, we must accommodate the effect of the changing cross sectional area over the length of the bar on both the stresses and displacement.

4.5.1 <u>Normal Stresses in a Tapered Bar</u>

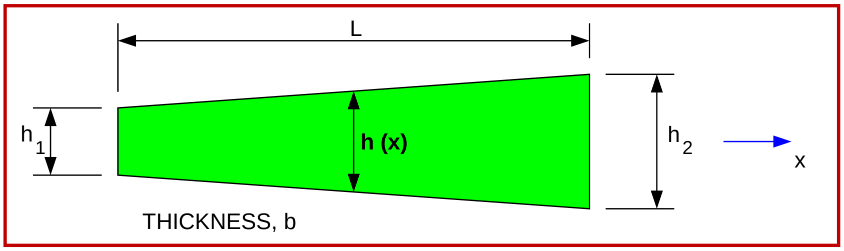
The normal stresses that occur in a tapered bar or rod are determined using Eq. (3.4). The only consideration made to account for the taper in the bar is to adjust the area A to correspond with its position along the length of the bar. Let's consider the tapered bar presented in Fig. 4.5. The thickness of the bar, b, is constant along its entire length. The height, h_x, of the bar varies with position x according to the relation:

$$h_x = h_1 + (h_2 - h_1)(x/L) \qquad \text{(a)}$$

The area, A_x, at any position x along the length of the bar is then:

$$A_x = bh_x = b[h_1 + (h_2 - h_1)(x/L)] \qquad (4.9)$$

Fig. 4.5 Geometry of a tapered bar with uniform thickness.



The normal stresses σ_x due to an axial force P are determined by substituting Eq. (4.9) into Eq. (3.4) to obtain:

$$\sigma_x = \frac{P}{A_x} = \frac{P}{b\left[h_1 + (h_2 - h_1)(x/L)\right]} \qquad (4.10)$$

where L is the length of the tapered bar.

EXAMPLE 4.13

For a tapered bar similar to the one shown in Fig. 4.5, determine the normal stress as a function of position, x along the length of the bar. The geometry of the bar is given by h_1 = 1.5 in., h_2 = 4 in., b = 1.25 in. and L = 48 in. The axial load imposed on the tapered bar is 11,000 lb.

Solution:

We employ Eq. (4.10) and write:

$$\sigma_x = \frac{P}{A_x} = \frac{P}{b\left[h_1 + (h_2 - h_1)(x/L)\right]} = \frac{11{,}000}{1.25\left[1.5 + (4 - 1.5)(x/48)\right]} \qquad \text{(a)}$$

$$\sigma_x = (422{,}400)/[72 + (2.5)x] \qquad \text{(b)}$$

The stress σ_x = 5,867 psi is a maximum at x = 0, and decreases to a minimum value of σ_x = 2,200 psi when x = 48 in. Note that the largest stress occurs at the smallest cross sectional area.

4.5.2 Deflection of Tapered Bars

The axial deflection of a tapered rod may be determined from Eq. (3.2) although we must modify the relation to accommodate for the changing cross sectional area over the length of the bar. This accommodation is more involved, because the total deformation of the bar is the sum of the incremental deflections at each position x along the entire length of the bar. Begin by considering an incremental length dx at some position x as shown in Fig. 4.6.

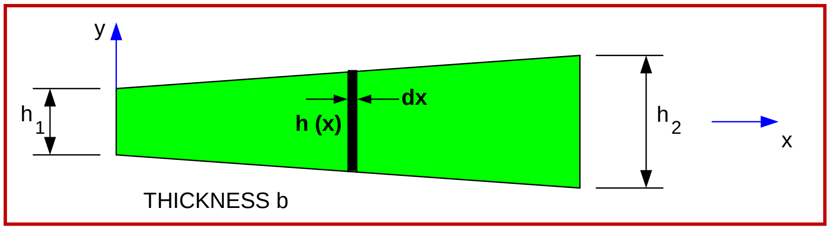


Fig. 4.6 An incremental length (slice) dx at position x along the length of the tapered bar.

To accommodate for the taper in the bar, determine the incremental deflection dδ of a bar of length dx. Because dx approaches zero, we treat its cross sectional area as a constant over the incremental length. Accordingly, we modify Eq. (3.2) to read:

$$d\delta = P\,dx/(A_x\,E) \tag{4.11}$$

where A_x is the cross sectional area of the bar at position x, which is given in Eq. (4.9).

Substitute Eq. (4.9) into Eq. (4.11) and simplify to obtain:

$$\delta = \frac{PL}{Eb}\int_0^L \frac{dx}{h_1 L + (h_2 - h_1)x} \qquad \text{(a)}$$

Integrating Eq. (a) yields:

$$\delta = \frac{PL}{Eb}\left(\frac{1}{h_2 - h_1}\right)\ln\left(\frac{h_2}{h_1}\right) \tag{4.12}$$

EXAMPLE 4.14

Determine the deflection of the tapered bar described in Example 4.13, if the bar is fabricated from a steel alloy with a modulus of elasticity $E = 30.0 \times 10^6$ psi.

Solution:

Let's solve this example problem by recalling Eq. (4.12).

$$\delta = \frac{PL}{Eb}\left(\frac{1}{h_2 - h_1}\right)\ln\left(\frac{h_2}{h_1}\right)$$

Substituting the parameters describing the geometry, the force and the material constant for the bar into this relation yields:

$$\delta = \frac{(11,000)(48)}{(30\times10^{6})(1.25)}\left(\frac{1}{4-1.5}\right)\ln\left(\frac{4}{1.5}\right) = 5.524\times10^{-3}\ \text{in.} \qquad \text{(a)}$$

Examine the magnitude of the deflection of the tapered bar. Is the deflection large or small? What reference do you use to judge? Clearly, the axial extension of the bar is small. For small quantities, we sometimes use the human hair as a reference. The diameter of a single strand of hair is about 2.5×10^{-3} in.; hence, the extension is about twice that diameter.

4.6 STRESS CONCENTRATION FACTORS

Let's consider the bar with a centrally located circular hole subjected to an axial tension force, as shown in Fig. 4.7.

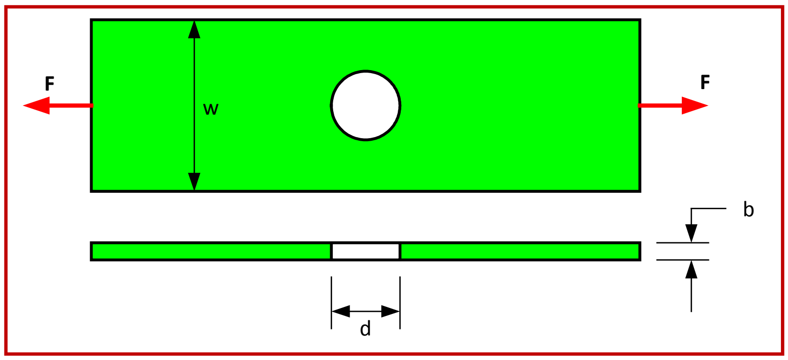


Fig. 4.7 A centrally located circular hole in an axially loaded bar.

The stress distribution in a section removed three or more diameters from the hole is nearly uniform, with a magnitude given by σ_o = F/(bw). However, on the section through the center of the hole, the stress distribution shows significant variation. The stresses increase markedly adjacent to the discontinuity (the hole) and concentrate at this location. The maximum value of the normal stress occurs adjacent to the hole, as indicated in Fig 4.8.

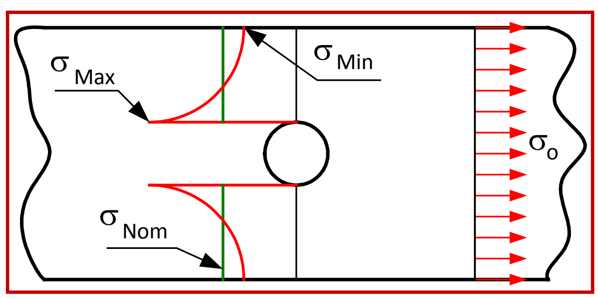


Fig. 4.8 Distribution of stress across a section through the center of the hole shows the concentration of stresses adjacent to the boundary of the hole.

We are interested in determining the maximum stress, σ_{Max} adjacent to the hole. It is convenient to express the maximum stresses in terms of a stress concentration factor by employing:

$$\sigma_{Max} = K\,\sigma_{Nom} \qquad (4.13)$$

where K is the stress concentration factor and σ_{Nom} is the nominal stress.

The **nominal stress** is the average stress across the net section containing the hole, and is given by:

$$\sigma_{Nom} = F/A_{Nom} = F/[(w - d)b] \qquad (4.14)$$

where b is the thickness of the bar, w is the bar width and d is the hole diameter.

The uniform stress σ_o and the nominal stress σ_{Nom} are related by:

$$F = \sigma_{Nom}\, A_{Nom} = \sigma_o\, A_{Uniform} \qquad (a)$$

Substituting for the areas in Eq. (a) and simplifying yields:

$$\sigma_{Nom} = [w/(w - d)]\sigma_o \qquad (4.15)$$

The nominal stress σ_{Nom} is always greater than the uniform stress σ_0, because the factor w/(w – d) is always greater than one.

The **stress concentration factor** K for a uniform thickness bar with a central circular hole subjected to axial loading is a function of the ratio of d/w, as shown in Fig. 4.9.

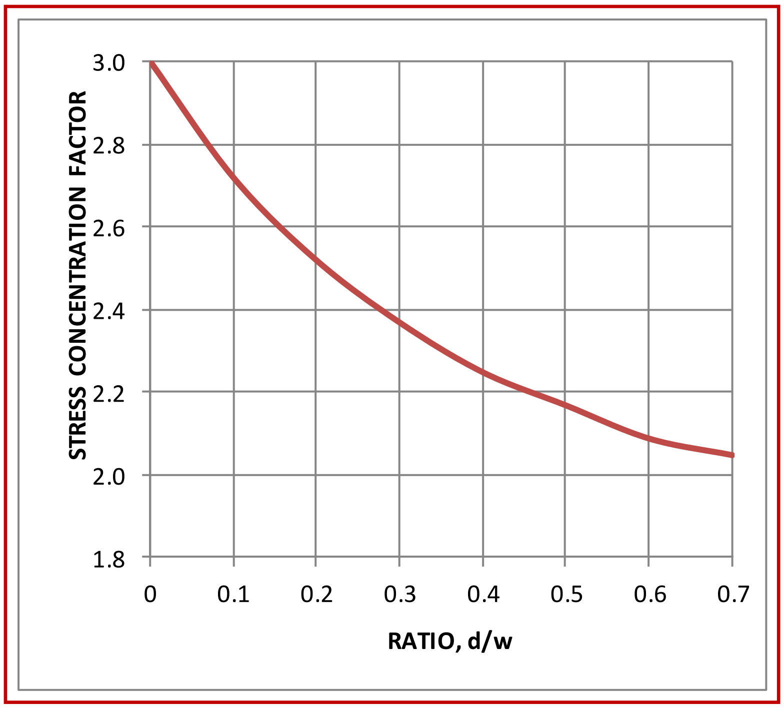


Fig. 4.9 Stress concentration factor for a central circular hole in an axially loaded bar.

EXAMPLE 4.15

A centrally located hole of diameter d = 0.60 in. is drilled in a long thin bar that is subjected to an axial load of 15,000 lb. If the bar is defined by w = 3.0 in., and b = 0.75 in., determine the nominal stress, the stress concentration and the maximum stress.

Solution:

The nominal stress is given by Eq. (4.14) as:

$$\sigma_{Nom} = F/[b(w - d)] = (15{,}000)/[(3.00 - 0.60)(0.75)] = 8{,}333 \text{ psi} \qquad \text{(a)}$$

The stress concentration K is determined from Fig. 4.9, by locating the intercept of a vertical line originating at d/w = 0.60/3.0 = 0.20 with the curve. This intercept gives K = 2.52.

Finally, the maximum stress is given by Eq. (4.13) as:

$$\sigma_{Max} = K\, \sigma_{Nom} = (2.52)(8{,}333) = 21.0 \text{ ksi} \qquad \text{(b)}$$

By drilling a hole in the bar, we have increased the maximum stress significantly. By using a stress concentration factor, we have a simple yet effective approach for solving a very difficult stress analysis problem. Also, you should be aware that the uniform stress in the bar prior to drilling the hole was σ_o = 6.667 ksi; therefore, the presence of the hole increased the stresses by a factor of 3.15.

4.7 SCALE MODELS

We have described methods for determining the stresses, strains and deflection of bars and rods subjected to axial tension or compression forces. The analysis required to determine these quantities is relatively simple, when the bar can be isolated and the forces acting on it are known. However, if the bar is part of a complex structure, determining the forces acting on the bar may prove to be difficult. In some cases, scale models of the structure under consideration are constructed and then subjected to scaled loads to verify structural integrity.

Geometric Scale Factor

Scale models of structures are usually much smaller than the real thing. Let's call the real structure — a bridge, building or stadium — the prototype, and consider a scale model that is 100 times smaller than the prototype. In this case, the geometric scale factor $\mathcal{S}$ = 1/100. If the same scale factor $\mathcal{S}$, is used for all three dimensions (i.e., length, width and thickness), we may write:

$$L_m = \mathcal{S}\, L_p$$

$$w_m = \mathcal{S}\, w_p \qquad (4.16)$$

$$b_m = \mathcal{S}\, b_p$$

where L, w and b represent length, width and thickness, and subscripts m and p refer to the model and the prototype respectively.

We note from Eq. (4.16), that the scale model is exactly the same shape as the prototype, except for its size. It is geometrically proportional.

Scaling Factor for Stresses

Let's continue with the concept of scaling and consider the stresses occurring in both the model and the prototype. The stress σ in a bar subjected to axial loading is given by Eq. (3.4) as:

$$\sigma = P/A = P/(w\ b) \qquad \text{(a)}$$

where P is the axial load applied to the bar and A = w b is the cross sectional area of that bar with width w and thickness b.

If the left side of Eq. (a) is divided by its right side, a unit dimensionless ratio involving the stress and its controlling parameters is formed:

$$\sigma(w\ b)/P = 1 \qquad \text{(b)}$$

Designate the model and prototype in Eq. (b) by writing:

$$\sigma_m\ (w_m\ b_m)/P_m = \sigma_p\ (w_p\ b_p\ )/P_p \qquad \text{(c)}$$

Rearranging the terms in Eq. (c) gives:

$$\sigma_m = (P_m\ /\ P_p)(w_p\ /\ w_m)(b_p\ /\ b_m)\sigma_p \qquad \text{(d)}$$

Substituting Eq. (4.16) into Eq. (d) yields:

$$\sigma_m = (P_m/\ P_p)(1/\mathbf{S}^2)\sigma_p \qquad \text{(e)}$$

Examining Eq. (e) shows that the stresses in the model and the prototype are related by the geometric scale factor $\mathbf{S}$ and load scale factor $\mathbf{L}$, which is defined by:

$$P_m = \mathbf{L}\ P_p \qquad (4.17)$$

Finally, we substitute Eq. (4.17) into Eq. (e) and obtain:

$$\sigma_m = (\mathbf{L}\ /\ \mathbf{S}^2)\ \sigma_p \qquad (4.18)$$

Clearly, the stresses produced in the model are related to the scale factors for the load and the geometry. It is also important to recognize that the geometric scale factor $\mathbf{S}$ usually differs significantly from the load scale factor $\mathbf{L}$.

EXAMPLE 4.16

Suppose we construct a scale model of the Golden Gate Bridge using a scale factor $\mathbb{S}$ of 1/750. Determine the size of the model if:

- Total bridge length ⇒ 8,981 feet
- Length of suspended structure ⇒ 6,450 feet
- Length of main span ⇒ 4,200 feet
- Length of each side span ⇒ 1,125 feet
- Width of bridge ⇒ 100 feet
- Diameter of main cable ⇒ 36.375 in.
- Width of roadway between curbs ⇒ 72 feet
- Lanes of vehicular traffic ⇒ 6
- Weight of main span per lineal foot ⇒ 21,300 lb
- Live load capacity per lineal foot ⇒ 4,000 lb

Solution:

The model bridge is constructed in three parts including the main span and the two side spans. The length of the main span of the model is given by Eq. (4.16) as:

$$L_m = \mathbb{S}\, L_p = (1/750)(4{,}200) = 5.6 \text{ ft} = 67.2 \text{ in.}$$

The width of the model bridge is determined in the same manner as:

$$w_m = \mathbb{S}\, w_p = (1/750)(100) = 0.1333 \text{ ft} = 1.60 \text{ in.}$$

The diameter, D of the main cable on the model is:

$$D_m = \mathbb{S}\, D_p = (1/750)(36.375) = 0.0485 \text{ in.}$$

All the geometric features of the model of the bridge are determined in this manner.

EXAMPLE 4.17

Let's suppose a very small strain gage is installed on an alloy steel bar of a model with a geometric scale factor L_m/L_p = 1/900, which is used to simulate a critical structural member in a prototype. The gage provides a measurement of the strain equal to $1{,}575 \times 10^{-6}$ when the model bridge was fully loaded. If the scaling factor for the load $\mathbb{L}$ = 1/65,000, determine the stress in the main cable of the prototype.

Solution:

The stress acting on the bar in the model is given by Hooke's law as:

$$\sigma = E\, \varepsilon \qquad (3.5)$$

Substituting numerical values for the modulus of elasticity and the strain into Eq. (3.5) gives the stress in the main cable of the model as:

$$\sigma_m = E_m\, \varepsilon_m = (30 \times 10^6)(1{,}575 \times 10^{-6}) = 47{,}250 \text{ psi} \qquad \text{(a)}$$

where the modulus of elasticity $E = 30 \times 10^6$ psi for steel.

Solving Eq. (4.18) for the stress on the prototype gives:

$$\sigma_p = (\mathbf{S}^2/\ \mathbf{L})\, \sigma_m \qquad \text{(b)}$$

Substituting σ_m and the numerical scale factors for $\mathbf{S}$ and $\mathbf{L}$ into Eq. (b) yields:

$$\sigma_p = \left(\frac{\mathbf{S}^2}{\mathbf{L}}\right)\sigma_m = \left(\frac{(1/900)^2}{(1/65{,}000)}\right)(47{,}250) = (0.08025)(47{,}250) = 3{,}792 \text{ psi}$$

This example illustrates that the scale factors for the load and the geometry of the structure should be selected to limit the stresses induced in the model. In most instances, the scale factor for the load is much smaller than the scale factor for the geometry.

Scaling Factor for Displacements

A model of the structure also provides an opportunity for displacement measurements that can be used to predict displacements in the prototype. To develop the displacement relation between the model and the prototype, we again seek a unit dimensionless quantity that includes the variables controlling the displacement. From Eq. (3.2) it is evident that the displacement δ of a rod of length L subjected to an axial load P is given by:

$$\delta = PL/AE = PL/[(w\ b)E] \qquad \text{(a)}$$

If the left side of Eq. (a) is divided by its right side, a unit dimensionless ratio is formed:

$$[\delta(w\ b)E]/(PL) = 1 \qquad \text{(b)}$$

Identify the model and the prototype with appropriate subscripts in Eq. (b) and write:

$$\frac{\delta_m (w_m b_m) E_m}{P_m L_m} = \frac{\delta_p (w_p b_p) E_p}{P_p L_p} \qquad \text{(c)}$$

Solving Eq. (c) for δ_m and substituting Eqs. (4.16) and (4.17) into the result, yields:

$$\delta_m = [\mathbf{L} /(\mathbf{S}\ \ \mathbf{E})]\ \delta_p \qquad (4.19)$$

where is the modulus scale factor between the model and the prototype that is given by:.

$$\mathbf{E} = E_m / E_p \qquad (4.20)$$

If the same materials are employed in the manufacture of both the model and the prototype the modulus scale factor is one. However, selection of the model materials is not restricted. We may use a wide variety of materials to fabricate the model providing they respond in a linear elastic manner.

EXAMPLE 4.18

Suppose that a model of a truss type bridge structure is fabricated from members formed from sheet aluminum. The prototype structure is to be fabricated from steel with a span of 650 feet. The model is geometrically scaled so that its span is 4.5 ft. The capacity of the live load on the prototype is 18,000 lb/ft, and the model is loaded with 32 lb/ft. If the model deflects a distance of 1.9 in. under full load at the center of the span, determine the deflection of the prototype under the design load.

Solution:

We can rearrange Eq. (4.19) and write:

$$\delta_p = [(\mathbf{S}\,\mathbf{E})/\,\mathbf{L}]\delta_m \qquad \text{(a)}$$

Note the scale factors are determined from Eqs. (4.16), (4.17), and (4.20) as:

$$\mathbf{S} = L_m/L_p = (4.5)/(650) = 0.006923$$

$$\mathbf{L} = P_m/P_p = (32)/(18{,}000) = 1.778 \times 10^{-3} \qquad \text{(b)}$$

$$\mathbf{E} = E_m/E_p = (10.4 \times 10^6)/(30 \times 10^6) = 0.3467$$

Substituting the scale factors into Eq. (a) yields:

$$\delta_p = [(0.006923)(0.3467)/(1.778 \times 10^{-3})](1.9) = 2.565 \text{ in.}$$

In this instance, the deflection of the prototype is nearly equal to the deflection of the model. The choice of materials for the model and the scaling factors for both the load and geometry influence the differences in the stress and deflection between the model and the prototype.

4.8 SUMMARY

This chapter introduces methods for designing uniaxial members, which are long and thin. Flexible members, such as wire and cable, only support axial tensile loads. They cannot support compressive forces because they buckle under very small loads. Increasing tension forces applied along the axis of these wires and cables results in deformation until they yield and finally break or rupture.

Rods and bars are sufficiently stiff to carry compressive forces. In this chapter, we consider only axial loading of these structural members. The case of transverse loading is covered later in the course on Mechanics of Materials. The equations derived for stresses and deflections in wire and cable are also applicable to bars and rods. When dealing with compressive forces and stresses, a minus sign is used to indicate if the loading is compressive. It is assumed that compressively loaded bars and rods are sized so they will not fail by buckling. Examples are presented for determining stresses and deflections of uniform axial bars.

The equations developed for uniform members subjected to axial forces are summarized below:

$$\mathbf{SF_u} = S_u / \sigma_{design} \qquad (4.1)$$

$$\mathbf{SF_y = S_y / \sigma_{design}} \qquad (4.2)$$

$$\sigma = P_{Design} / A \qquad (4.3)$$

$$S_{ys} = 0.5774\ S_y \qquad (4.4)$$

$$\mathbf{SF_{ys}} = S_y / \tau_{Design} \qquad (4.5)$$

The concept of stresses on section cuts that produce FBDs with both normal and oblique planes was introduced. Stresses on a section with an inclined cut at an angle θ were determined using the equilibrium relations and the definition for normal and shear stresses as:

$$\sigma_\theta = (\sigma_x) \cos^2 \theta \qquad (4.6)$$

$$\tau_\theta = (\sigma_x/2) \sin 2\theta \qquad (4.7)$$

In Fig. 4.3, we note that both the normal stress σ_θ and the shear stress τ_θ vary with the angle θ. The normal stress is a maximum when θ = 0°, and the section cut is perpendicular to the axis of the bar. The shear stress vanishes on this section. The shear stress is a maximum when θ = 45°. On this plane, $\sigma_{45} = \tau_{45} = \sigma_x /2$.

Uniaxial members with steps along their length were introduced and equations for determining the stresses and deformation at each portion of the bar were developed. The stress is easily computed from Eq. (3.4) by using the internal force and cross sectional area for each section of the bar. Displacements of each section of the bar are determined from Eq. (3.2) and then superimposed to give the total deflection as:

$$\delta_{total} = \delta_1 + \delta_2 + \delta_3 + \ldots.. + \delta_n \qquad (4.8)$$

Tapered bars were also considered, where the cross section of the bar changes as a function of position along the length of the bar. The stresses in the tapered bar vary with position along the length and are given by:

$$\sigma_x = \frac{P}{A_x} = \frac{P}{b\left[h_1 + \left(h_2 - h_1\right)\left(x/L\right)\right]} \qquad (4.10)$$

For deflection of the tapered bar, we considered an incremental length dx along the length of the bar and modified Eq. (3.2) to read as:

$$d\delta = P\, dx/(A_x\, E) \qquad (4.11)$$

To determine the total deflection δ of the tapered bar, integrate Eq. (4.11) from zero to L to obtain:

$$\delta = \frac{PL}{Eb}\left(\frac{1}{h_2 - h_1}\right)\ln\left(\frac{h_2}{h_1}\right) \qquad (4.12)$$

The concept of stress concentrations due to geometric discontinuities in uniaxial members was introduced. We showed a method for determining the maximum stress, σ_{Max} adjacent to a centrally located circular hole. We expressed the maximum stresses in terms of a stress concentration factor K by employing:

$$\boxed{\sigma_{Max} = K\,\sigma_{Nom} \qquad (4.13)}$$

The stress concentration factor for a uniform thickness bar with a central circular hole subjected to axial loading is a function of the ratio of d/w, was shown in Fig. 4.9.

The nominal stress σ_{Nom} is the average stress across the net section of the member containing the hole, and is given by:

$$\boxed{\sigma_{Nom} = F/A_{Nom} = F/[(w - d)b] \qquad (4.14)}$$

The use of structural models that are geometrically similar to actual structures was described. The stresses and displacements produced by loading or deforming a model are related to those developed in the prototype (structure). Relations involving scaling factors are employed to determine the stresses and displacements in the structure based on measurements made on a geometrically scaled model. The important equations for scale modeling are given below:

$$\boxed{\begin{aligned} L_m &= \mathbf{S}\,L_p \\ w_m &= \mathbf{S}\,w_p \qquad (4.16) \\ b_m &= \mathbf{S}\,b_p \end{aligned}}$$

$$\boxed{P_m = \mathbf{L}\,P_p \qquad (4.17)}$$

$$\boxed{\sigma_m = (\mathbf{L} / \mathbf{S}^2)\,\sigma_p \qquad (4.18)}$$

$$\boxed{\delta_m = [\mathbf{L} /(\mathbf{S}\,\mathbf{E})]\,\delta_p \qquad (4.19)}$$

$$\boxed{\mathbf{E} = E_m /E_p \qquad (4.20)}$$

CHAPTER 5

TRUSSES

5.1 INTRODUCTION

Trusses are structures commonly used to efficiently span relatively long distances. As such, they are employed in the design of long bridges, roofs of stadiums and buildings both large and small. A **truss** is defined as a large structure made of many smaller uniaxial members (bars, rods or cables) that are connected together to form a strong and rigid arrangement. The term truss is derived from the Middle English word **trusse**, which means, "bundle". Several of the classical geometries used in truss design for bridges and roofs are presented in Fig. 5.1.

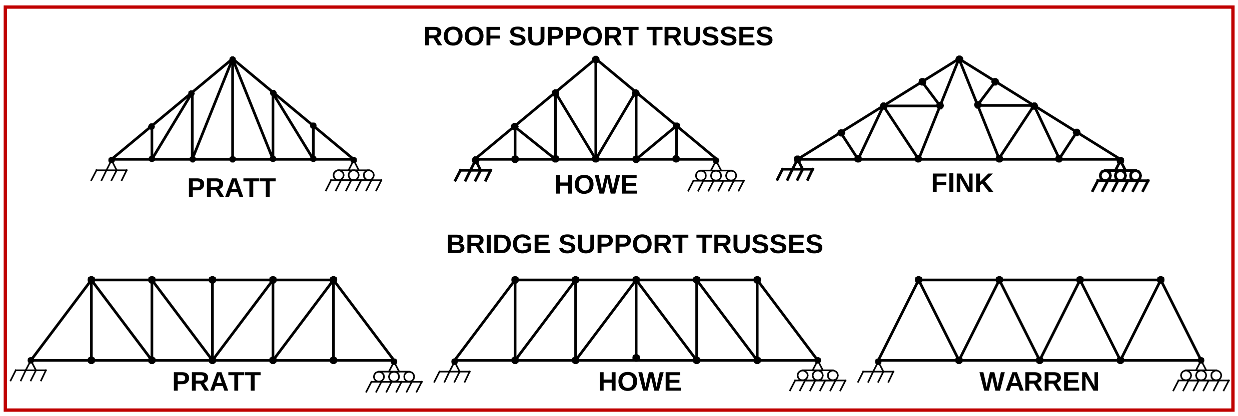


Fig. 5.1 Different types of trusses used in the design of bridges and roof supports.

5.1.1 Stability

If we examine the geometry of the trusses presented in Fig. 5.1, it is clear that they are all made from straight members arranged to form an array of triangles. The triangular form is very important, because it insures stability of the truss regardless of the direction of the applied forces. Let's examine the stability of a four-member frame with a rectangular arrangement of the members as defined in Fig. 5.2.

The four-bar-rectangular frame in Fig. 5.2a is fabricated with joints at the four corners A, B, C and D. If a horizontal force F is applied to this frame at point C, the joints tend to act as pins[1] and the frame rotates, as shown in Fig. 5.2b. If the force F is maintained as the frame rotates, the structure will collapse. To stabilize the rectangular frame of Fig. 5.2, we add a fifth bar between points A and D as shown in Fig. 5.3. With the application of the horizontal force F, the joints will again tend to rotate. However, significant rotation of the structure requires member AD to elongate by yielding. If member AD is designed with sufficient strength, the truss cannot deform and its stability is ensured.

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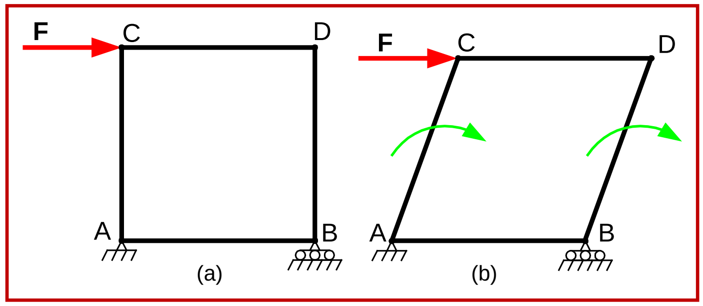
[1]Joints can be designed to resist rotation; however, it is usually more efficient and less costly to insure stability by adding an additional member to the structure that prevents rotation.

Fig. 5.2 A four-bar frame with a rectangular arrangement is unstable.



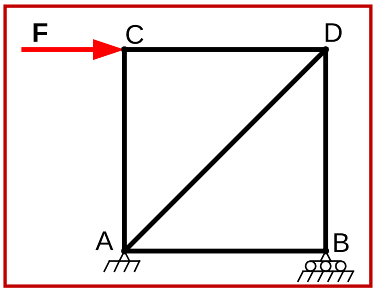


Fig. 5.3 Converting the rectangular structure into a truss with triangular elements assures its stability.

Note the addition of the member AD has converted the rectangular truss arrangement, shown in Fig. 5.3, into one consisting of two triangles. The triangular element is the essence of stability of a structural arrangement. A simple three-member triangular truss is illustrated in Fig. 5.4, with a horizontal force F applied at joint C. Even if the joints are made with frictionless pins (free to rotate), the structure remains stable. Stability is insured by the presence of member AC that must experience large deformations before a significant rotation of member BC can occur.

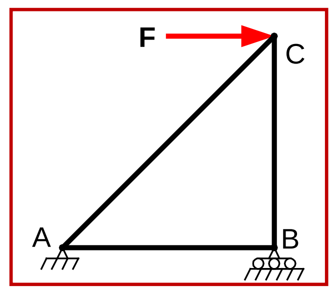


Fig. 5.4 Triangular structural elements are stable.

Examination of the truss geometries, presented in Fig. 5.1, shows the repeated use of the triangular elements to increase the span of the truss. With the addition of the new members, the number of joints used in fabricating the truss is also increased. For trusses fabricated from simple triangular elements, the number of joints and the number of members are related by:

$$n = 2k - 3 \tag{5.1}$$

where n is the number of members and k is the number of joints.

5.1.2 Truss Members

Members of a truss are long and thin. As such, they cannot support significant transverse (lateral) forces; consequently, loads applied to the structure must be placed at the joints. The weight of most truss members is usually negligible in comparison to the applied forces. However, if the weight of a specific member is significant, because of extremely long length, the weight of the member is divided by two and that amount is added to its two joints at the ends — thus accounting for the effect of the member's weight in the analysis.

Joints are often constructed using gusset plates to effect the connection of the various members that meet at a specified point. A gusset plate used in fabricating a joint between four members is illustrated in Fig. 5.5. The truss members are fastened to the gusset plate by riveting, bolting or welding.

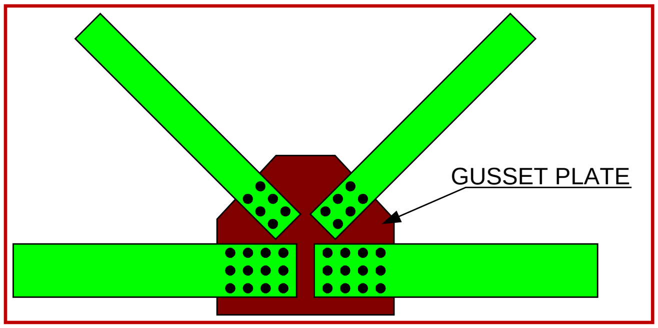


Fig. 5.5 A joint formed with a gusset plate at the junction of four truss members.

Even though the joint between two or more members may resist rotation and support moments, we assume that the joint is free to rotate. This assumption does not produce significant errors in the analysis if the individual truss members are long and flexible. Also this assumption permits us to treat each truss member as a **two-force member** — free of moments and transverse forces. For a two-force member, the force acting on the member must be transmitted through the axis of the member. Consequently, the direction of the force is established by the orientation of the member. The fact that the individual truss members are two-force members enables us to represent members, joints, and sections with three different types of FBDs, as shown in Fig. 5.6.

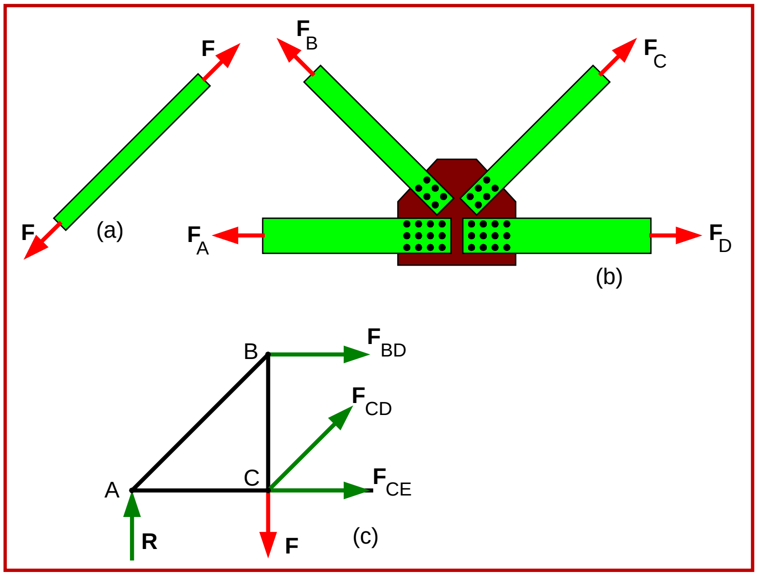


Fig. 5.6 FBDs for a truss:
(a) A single truss member
(b) a truss joint
(c) a truss section.

5.2 METHOD OF JOINTS

In performing a design analysis of a truss, we employ either the method of joints or the method of sections. Both methods employ a FBD of a portion of the truss, as illustrated in Fig. 5.6. With the method of joints, we "remove" a specified joint from the truss and draw a free diagram of that joint including both the internal and external forces that act on it. With the method of sections, we make a section cut to remove either the left or right side of the truss. We then prepare a FBD of one side or the other of the sectioned truss.

The method of joints is an extension of the material previously described in Chapters 2, 3 and 4. The analysis incorporates six steps:

1. Draw a FBD of the entire truss structure.
2. Apply the equations of equilibrium to solve for the reactions at the supports.
3. Select a joint and construct a FBD that includes both the internal and external forces.
4. Apply the equations of equilibrium to solve for the forces in the members that are connected at this joint. We assume that the joint acts like a particle; hence, it is represented as a point and equilibrium is satisfied if $\Sigma\mathbf{F} = 0$. It is not necessary to consider $\sum\mathbf{M} = 0$.
5. Determine the stresses in the individual members by using $\sigma = P/A$, and note if the stresses are tensile or compressive.
6. Compare the stress in each member with the strength of the material used to fabricate the truss and establish the margin of safety for each member.

To illustrate the procedure for a design analysis of a simple truss, consider the following example.

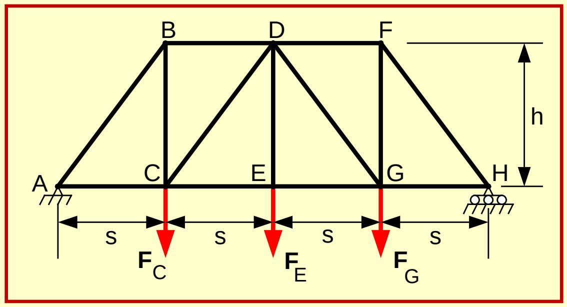
EXAMPLE 5.1

Consider the Howe truss shown in Fig. E5.1. The total span of the truss is $S_p= 4s = (4)(15) = 60$ ft, and the height h = 20 ft. External forces F_C, F_E and F_G are applied at joints C, E and G. Determine the margin of safety for truss members AB and AC[2]. The following parameters are necessary for a numerical solution for the margin of safety.

The forces are $F_C = 40$ kip, $F_E = 60$ kip, and $F_G = 40$ kip

The cross sectional areas of the bars employed in the construction of the truss are $A_{AC} = 5$ in^2. and $A_{AB} = 8$ in^2. These bars are fabricated from hot-rolled, mild-steel with a yield strength of 40 ksi.

Fig. E5.1

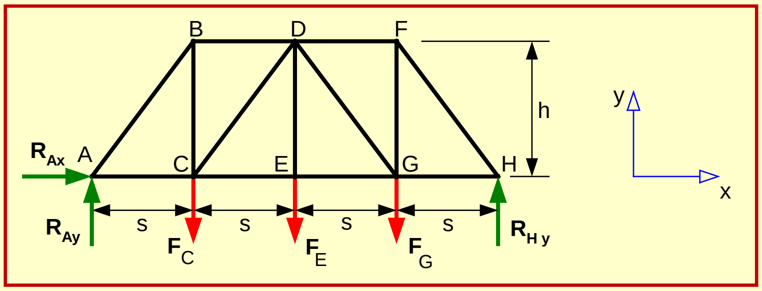


Solution:

Step 1: Draw a FBD of the entire truss structure.

To determine the reactions at the left and right hand supports of the truss, the supports are removed and replaced with reaction forces R_{Ay}, R_{Hy} and R_{Ax} in the FBD, as shown In Fig. E5.1a:

Fig E 5.1a



[2] AB and AC refer to members in the truss that extend from point A to B and point A to C respectively.

Step 2: Apply the equations of equilibrium to solve for the reactions at the supports. It is immediately evident from $\sum F_x = 0$ that the reaction force at the left support $R_{Ax} = 0$.

Consider the sum of the moments about point A and write:

$$\Sigma M_A = (4s)R_{Hy} - (3s)F_G - (2s)F_E - (s)F_C = 0 \qquad (a)$$

Solving Eq. (a) for R_{Hy} yields:

$$R_{Hy} = (1/4)(3F_G + 2F_E + F_C) = (1/4)(120 + 120 + 40) = 70.0 \text{ kip} \qquad (b)$$

From equilibrium we write:

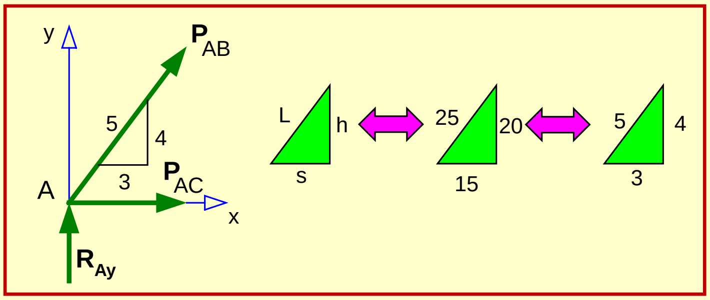
$$\Sigma F_y = R_{Ay} + R_{Hy} - F_C - F_E - F_G = 0$$

$$R_{Ay} = 40 + 60 + 40 - 70 = 70.0 \text{ kip} \qquad (c)$$

Both R_{Ay} and R_{Hy} are equal to 70.0 kip. The equality of the two reactions is expected because the geometry and the loading of the truss are both symmetrical.

Step 3: Select a joint and construct a FBD that includes both the internal and external forces acting at that joint. We select the joint at point A, because members AB and AC, the subject of the analysis, meet at this joint. The FBD is presented in Fig. E5.1b:

Fig. E5.1b



Note, the dimensions of the span and height of the primary triangular element in the truss provide the orientation of the internal force P_{AB} in member AB. In this FBD we have used:

- The symbol P to represent internal forces.
- The symbol R to represent reaction (external) forces.
- Subscripts to identify the truss members.
- The internal forces P_{AB} and P_{AC} are shown as positive (tensile)[3].

Step 4: Apply the equations of equilibrium to solve for the forces in the members that connect at the subject joint. Because we have assumed that the joint acts like a pin, the joint is represented as a point and equilibrium is satisfied if **ΣF** = 0, or $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Let's begin with:

$$\Sigma F_y = R_{Ay} + (4/5)\, P_{AB} = 0 \qquad (d)$$

Solve this relation for P_{AB} and use the results of Eq. (c) to obtain:

[3]It is usual practice to assume that the unknown forces in truss members are tensile.

$$P_{AB} = -(5/4)\, R_{Ay} = -(5/4)(70.0) = -87.5 \text{ kip} \qquad \text{(e)}$$

The negative sign for P_{AB} indicates that it is a compressive force. Our initial assumption indicating this force was tensile (+) was in error. This error does not cause a problem — we note the sign, declare that member AB is in compression, and treat P_{AB} as a negative quantity in subsequent steps in the analysis.

Next, write the equilibrium relation for the force components in the x direction.

$$\Sigma F_x = (3/5)P_{AB} + P_{AC} = 0$$

$$P_{AC} = -(3/5)P_{AB} = -(3/5)(-87.5) = +52.5 \text{ kip} \qquad \text{(f)}$$

The plus sign for P_{AC} indicates that the internal force in truss member AC is tensile. Our assumption about the direction of the force P_{AC} in constructing the FBD for joint A was correct.

Step 5: Determine the stresses in the individual members of the truss from $\sigma = P/A$, and note if the stresses are tensile or compressive.

For member AB and AC, we determine the stress σ from:

$$\sigma_{AB} = P_{AB} / A_{AB} = -(87.5 \text{ kip})/(8 \text{ in}^2) = -10.94 \text{ ksi} \qquad \text{(g)}$$

$$\sigma_{AC} = P_{AC} / A_{AC} = +(52.5 \text{ kip})/(5 \text{ in}^2) = +10.50 \text{ ksi} \qquad \text{(h)}$$

The minus sign for σ_{AB} indicates that the stress in member AB is compressive[4]. The stress in member AC is tensile as indicated by the positive sign.

Step 6: Compare the stress in each member to the strength of the material used to fabricate the truss and then establish the margin of safety for the structure.

From Eq. (4.2), we determine the safety factor for members AB and AC as:

$$SF_{AB} = S_y/\sigma_{AB} = (40.0)/(10.94) = 3.656 \qquad \text{(i)}$$

$$SF_{AC} = S_y/\sigma_{AC} = (40.0)/(10.5) = 3.810 \qquad \text{(j)}$$

Finally, let's determine the margin of safety MOS according to its definition:

$$MOS = SF - 1 \qquad (5.2)$$

Then for members AB and AC, we determine:

$$MOS_{AB} = 3.656 - 1 = 2.656 = 265.6\% \qquad \text{(k)}$$

$$MOS_{AC} = 3.810 - 1 = 2.810 = 281.0\% \qquad \text{(l)}$$

[4] Because the stress in member AB is compressive, there is a possibility that this member may fail by buckling. Another analysis must be conducted to determine if the critical buckling load on this member has been exceeded. Methods for determining the critical buckling load on bars loaded in axial compression are covered in Chapter 7.

We have completed our solution. Now it is mandatory to interpret the results for the safety of both members AB and AC. These margins of safety appear to be reasonable. If the external loads are increased by as much as 100 percent, members AB and AC will not fail or begin to deform.

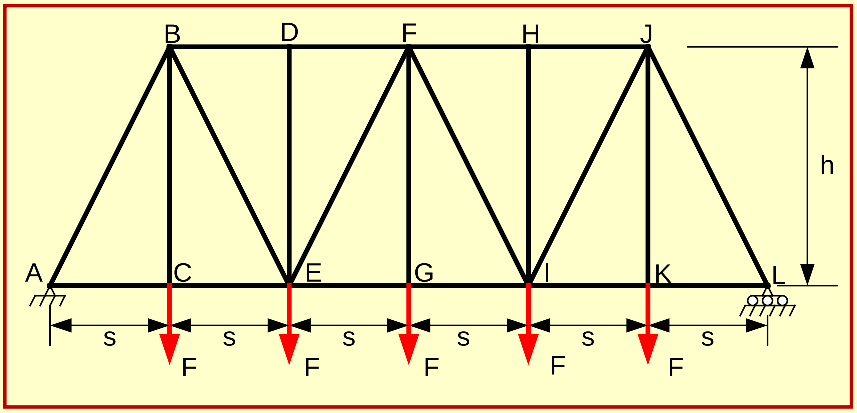
When designing structures, it is common to incorporate safety factors from 2 to 4 to provide margins of safety of 100 to 300%. These safety factors are to accommodate for uncertainties in the construction and operation over the life of the structure. Uncertainties include:

- A possible increase in loading, deliberate or accidental, any time during the life of the structure.
- Variations in the strength of the materials used in construction.
- Effect of fatigue (cyclic loading) in generating flaws (cracks) in the structure.
- Effects of corrosion on reducing section size or on inducing crack initiation in one or more members of the structure.

EXAMPLE 5.2

Determine the internal forces in members AB, AC, BC, BD, BE, DE, and DF of the truss defined in Fig. E5.2. Also determine the margin of safety for the highest stressed member. The external forces applied at the joints C, E, G, I, and K all equal to 75.0 kip. The cross sectional area of all of the members used to construct the truss is equal to 10 in^2. The material used in fabricating the truss is hot rolled steel with a yield strength of 44 ksi. Note, s = 15 ft and h = 30 ft.

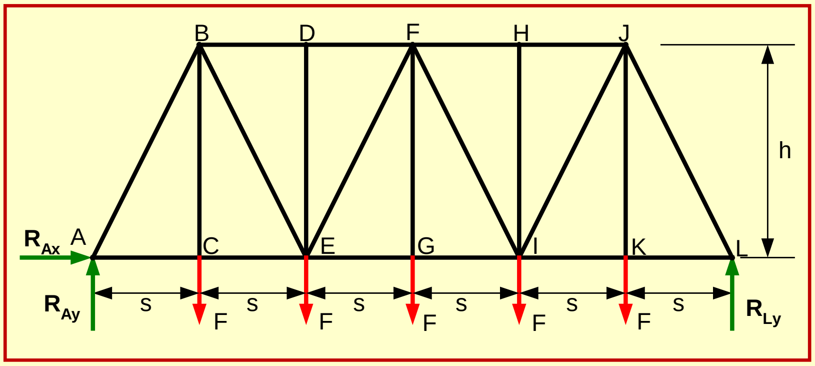
Fig. E5.2



Solution:

Step 1: Construct a FBD of the entire truss as shown in Fig. E5.2a:

Fig. E5.2a:



Step 2: Solve for the reactions at the supports.

Again it is clear from $\sum F_x = 0$ that $R_{Ax} = 0$.

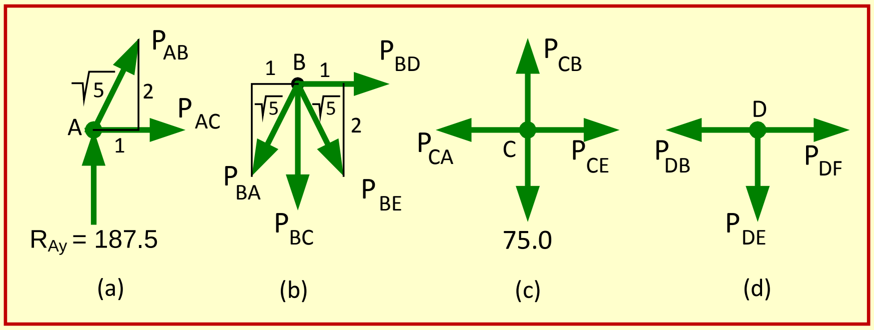
Inspection of the structure shows that it is symmetric with respect to both geometry and loading. For this reason, it is evident that $R_{Ay} = R_{Ly} = R$.

$$\Sigma F_y = 2R - 5F = 0$$

$$R = (5/2)(75) = 187.5 \text{ kip} \qquad \text{(a)}$$

Step 3: Select the joints necessary for solution of the unknown forces and prepare FBDs as shown in Fig. E5.2b.

Fig. E5.2b



We have selected joints A, B, C and D because the forces occurring at these joints in the FBDs correspond to the unknown forces listed in the problem statement. Note, we have assumed all of the internal forces to be tension, which is the usual practice. Also, the first subscript gives the letter of the joint described in the FBD. The second subscript gives the direction of the force. Because the truss members are two-force members in equilibrium, $P_{AB} = P_{BA}$, etc.

Step 4: Use equilibrium relations as necessary to solve for the unknown forces.

We have four FBDs — one each for joints A, B, C and D. It is possible to write a total of eight relevant equilibrium relations, if they are required. Let's write equilibrium equations beginning with the FBD of joint A, then B, C and D.

For joint A:

$$\Sigma F_y = R_{Ay} + [(2)/(\surd 5)]\, P_{AB} = 0$$

$$P_{AB} = -[(\surd 5)/(2)]R_{Ay} = -[(\surd 5)/(2)](187.5) = -\,209.6 \text{ kip} \qquad \text{(b)}$$

Note, the minus sign for P_{AB} indicates the internal force in member AB is compressive.

$$\Sigma F_x = P_{AC} + [(1)/(\surd 5)]\, P_{AB} = 0$$

$$P_{AC} = -\,[(1)/(\surd 5)]P_{AB} = -\,[(1)/(\surd 5)](-\,209.6) = 93.74 \text{ kip} \qquad \text{(c)}$$

For joint B:

Examining the FBD for joint B shows that four forces act at this point. We know only one of these forces, namely P_{BA}. The remaining three forces are unknown quantities. Because only two meaningful equilibrium relations may be written for a concurrent force system, the analysis of joint B is not possible with the methods presented at this time. The analysis of joint B is possible only if we reduce the number of unknown forces to two. With the reduction of forces at joint B as a motivation, let's move onto joint C.

For joint C:

$$\Sigma F_x = P_{CE} - P_{CA} = 0$$

$$P_{CE} = P_{CA} = P_{AC} = 93.74 \text{ kip} \qquad \text{(d)}$$

$$\Sigma F_y = P_{CB} - 75.0 = 0$$

$$P_{CB} = 75.0 \text{ kip} \qquad \text{(e)}$$

Consider joint B again:

From the analysis of Joint C, we determined that P_{CB} = 75 kip. Because P_{CB} and P_{AB} are known, only two unknown forces (P_{BE} and P_{BD}) act at joint B; therefore, we may proceed with the solution by writing equilibrium relations:

$$\Sigma F_y = -P_{BC} - [(2)/(\sqrt{5})](P_{BA} + P_{BE}) = 0$$

$$P_{BE} = -[(\sqrt{5})/(2)]P_{BC} - P_{BA} = [-(\sqrt{5})/(2)](75.0) - (-209.6) = +125.7 \text{ kip} \qquad \text{(f)}$$

$$\Sigma F_x = P_{BD} - [(1)/(\sqrt{5})](P_{BA} - P_{BE}) = 0$$

$$P_{BD} = (1/\sqrt{5})(P_{BA} - P_{BE}) = [(1)/(\sqrt{5})](-209.6 - 125.7) = -150.0 \text{ kip} \qquad \text{(g)}$$

Finally, consider joint D:

$$\Sigma F_y = P_{DE} = 0 \qquad \text{(h)}$$

$$\Sigma F_x = -P_{DB} + P_{DF} = 0$$

$$P_{DB} = P_{DF} = -150.0 \text{ kip} \qquad \text{(i)}$$

Because P_{DE} = 0, member DE of the truss structure is a zero-force member. We will discuss the arrangement of members at a joint in a truss that results in zero-force members later in this chapter. Let's summarize the results of the analysis in Table 5.1.

Table 5.1
Summary of forces and stresses determined in Example 5.2

MEMBER	FORCE kip	AREA in^2	STRESS ksi	SAFETY FACTOR	MARGIN OF SAFETY
AB	209.6 C	10	20.96 C	2.099	1.099
AC	93.74 T	10	9.374 T	4.694	3.694
BC	75.0 T	10	7.50 T	5.867	4.867
BD	150.0 C	10	15.00 C	2.933	1.933
BE	125.7 T	10	12.57 T	3.500	2.500
DE	0	10	0	∞	∞
DF	150.0 C	10	15.00 C	2.933	1.933

Step 5: Solve for the stresses.

Let's determine the stresses in member AB of the truss.

$$\sigma_{AB} = P_{AB} / A_{AB} = -(209.6 \times 10^3)/(10) = -20.96 \text{ ksi} \qquad \text{(j)}$$

The results for the stresses in the other members in the truss are shown in Table 5.1.

Step 6: Determine the safety factor and the margin of safety for the most highly stressed member.

Inspection of Table 5.1 indicates that member AB is the most highly stressed member in the truss. We determine its safety factor from:

$$SF = S_y/\sigma \qquad (4.2)$$

For member AB:

$$SF = S_y/\sigma = (44)/(20.69) = 2.099 \qquad \text{(k)}$$

$$MOS = SF - 1 = 2.099 - 1 = 1.099 = 109.9\% \qquad \text{(l)}$$

Finally, let's interpret the results that are presented in Table 5.1 from a designer's viewpoint.

First, we note a significant difference in the stresses from one member to another. It is clear that the choice of 10 in^2 for the cross sectional areas of all of the members was not appropriate. A more uniformly safe design would have been possible if the areas had been adjusted for the different forces imposed on the different members. Area adjustment is possible, but it is preferable to use structural members with standard section sizes. It is not usually economically feasible to specify sectional areas that differ from the areas available with standard size members.

Second, the safety factor and the margin of safety for member AB may be too low. We normally seek a safety factor of 2 to 4 to account for contingencies that have been described previously in Example 5.1.

Third, two **zero-force members** have been incorporated in the design: members DE and IH carry no force. The designer may have added them to the truss structure for aesthetic reasons, to provide backup if certain diagonal members failed by yielding, or in anticipation of loads not considered in this example. However, if a good reason for using these members does not exist, they should be eliminated. They add no strength to the structure and increase its cost, weight and construction time.

EXAMPLE 5.3

A scissors truss, illustrated in Fig. E5.3, is loaded with forces F_1, F_2 and F_3 at the joints located at points B, C and D, respectively. If the truss is fabricated from HR 1020 steel with a yield strength of 290 MPa, determine the size of members AB, AF, BC and BF. The safety factor is specified as 3.25. The forces and dimensions are:

F_1 = 20 kN, F_2 = 40 kN, F_3 = 50 kN ⇒ ⇒ h = 5 m, s = 8 m

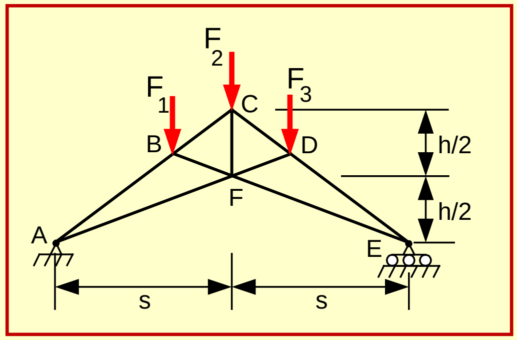


Fig. E5.3

Solution:

Step 1: Construct a FBD of the entire truss as indicated in Fig. E5.3a. We remove the supports and replace them with the reactions for a pivot and roller. However, a problem in dimensioning the FBD is encountered. The dimensions locating points B and D (the points of application of forces F_1 and F_3) are not given explicitly in Fig. E5.3. It will be necessary to consider the geometry of the truss and compute the x coordinate of these points. Begin by forming the triangle ADE, as shown in Fig. E5.3b. Remove the member EF to make the triangle more apparent.

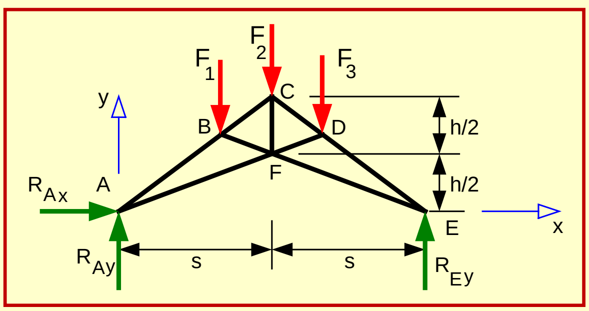


Fig. E5.3a

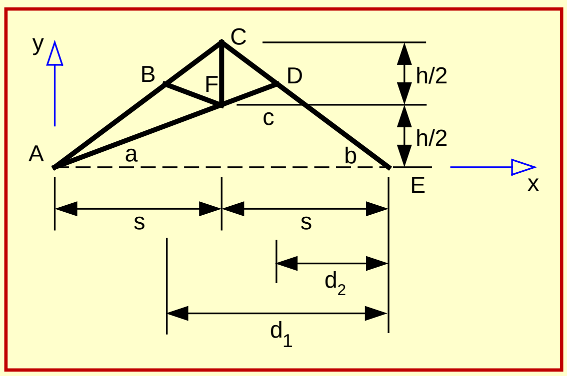


Fig. E5.3b

Solving for the angles a, b and c in the triangle ADE gives:

$$a = \tan^{-1}(h/2s) = \tan^{-1}(5/16) = 17.35°$$

$$b = \tan^{-1}(h/s) = \tan^{-1}(5/8) = 32.00° \qquad \text{(a)}$$

$$c = 180° – a – b = 180 – 17.35 – 32.00 = 130.65°$$

The law of sines is employed to write:

$$\frac{\sin(c)}{2s} = \frac{\sin(a)}{DE}$$

Solving this relation for DE gives:

$$DE = (2s)\frac{\sin(a)}{\sin(c)} = (16)\left(\frac{\sin(17.35)}{\sin(130.65)}\right) = 6.289 \text{ m} \qquad \text{(b)}$$

The distances d_2 and d_1 locating the points of application of forces F_3 and F_1 relative to point E are given by:

$$d_2 = DE\cos(b) = (6.289)\cos(32.00) = 5.333 \text{ m} \qquad \text{(c)}$$

Geometric symmetry enables us to write:

$$d_1 = 2s – d_2 = 16 – 5.333 = 10.667 \text{ m} \qquad \text{(d)}$$

Step 2: Solve for the reactions at the supports.

Again it is clear from $\sum F_x = 0$ that $R_{Ax} = 0$. (e)

Inspection of the structure shows that it is not symmetric with respect to the loading. For this reason, $R_{Ay} \neq R_{Ey}$. Let's consider moments about point E and write:

$$\Sigma M_E = – (2s)R_{Ay} + (d_1)F_1 + (s)F_2 + (d_2)F_3 = 0 \qquad \text{(f)}$$

Solving for R_{Ay} gives:

$$R_{Ay} = (d_1F_1 + sF_2 + d_2F_3)/(2s) = [(10.667)(20) + (8)(40) + (5.333)(50)]/(16) = 50.00 \text{ kN}$$

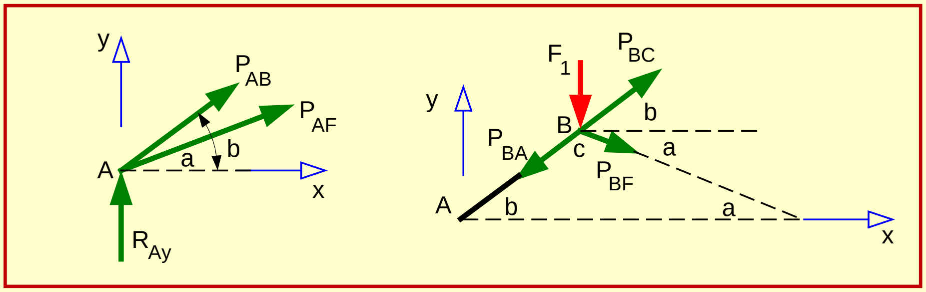
Let's solve for R_{Ey} by writing:

$$\Sigma F_y = R_{Ay} + R_{Ey} – F_1 – F_2 – F_3 = 0 \qquad \text{(g)}$$

$$R_{Ey} = 20 + 40 + 50 – 50.00 = 60.00 \text{ kN} \qquad \text{(h)}$$

Step 3: Select the joints necessary for solution of the unknown forces and prepare FBDs as shown in Fig. E5.3c.

Fig. E5.3c



We have selected joints A, and B because the forces occurring at these joints in the FBDs correspond to the unknown forces listed in the problem statement. Note, we have assumed all of the internal forces to be tension, which is the usual practice. Also, the first subscript gives the letter of the joint described in the FBD and the second subscript gives the direction of the force. Because the truss members are two-force members in equilibrium, $P_{AB} = P_{BA}$, etc.

Step 4: Use equilibrium relations as necessary to solve for the unknown forces.

We have two FBDs — one for joint A and another for joint B. It is possible to write a total of four relevant equilibrium relations if they are required. Let's begin to write equilibrium equations beginning with the FBD of joint A.

For joint A:

$$\Sigma F_y = R_{Ay} + P_{AB}\sin(b) + P_{AF}\sin(a) = 0$$

$$P_{AB} = -\left(\frac{R_{Ay} + P_{AF}\sin(a)}{\sin(b)}\right) = -\left(\frac{50.0 + P_{AF}\sin(17.35)}{\sin(32.00)}\right)$$

$$P_{AB} = -\ 94.35 - (0.5627)P_{AF} \qquad \text{(i)}$$

$$\Sigma F_x = P_{AB}\cos(b) + P_{AF}\cos(a) = 0$$

$$P_{AF} = -\left(\frac{P_{AB}\cos(b)}{\cos(a)}\right) = -\left(\frac{P_{AB}\cos(32.00)}{\cos(17.35)}\right) = -0.8885\,P_{AB} \qquad \text{(j)}$$

Combining the results from Eqs. (i) and (j) yields:

$$P_{AB} = -\ 188.7\ \text{kN} \qquad \Rightarrow \qquad \Rightarrow \qquad P_{AF} = 167.7\ \text{kN} \qquad \text{(k)}$$

Note, the minus sign for P_{AB} indicates the internal force in member AB is compressive.

For joint B:

Examining the FBD for joint B shows that four forces act at this point. We know two of these forces, namely P_{BA} and F_1. The remaining two forces are unknown quantities that are determined from the two equilibrium relations that apply for a concurrent force system.

$$\Sigma F_x = P_{BC}\cos(b) + P_{BF}\cos(a) - P_{BA}\cos(b) = 0$$

$$P_{BC}\cos(32.00) + P_{BF}\cos(17.35) - P_{BA}\cos(32.00) = 0$$

$$(0.8480)\,P_{BC} + (0.9545)\,P_{BF} - (-188.7)(0.8480) = 0$$

$$(0.8480)\,P_{BC} + (0.9545)\,P_{BF} + 160.0 = 0 \qquad \text{(l)}$$

$$\Sigma F_y = P_{BC}\sin(b) - P_{BF}\sin(a) - P_{BA}\sin(b) - F_1 = 0$$

$$P_{BC} \sin(32.00) - P_{BF} \sin(17.35) - P_{BA} \sin(32.00) - 20.0 = 0$$

$$(0.5299)\, P_{BC} - (0.2982)\, P_{BF} - (-188.7)(0.5299) - 20.0 = 0 \qquad \text{(m)}$$

Solving Eqs. (l) and (m) for the two unknowns gives:

$$P_{BF} = -22.33 \text{ kN} \qquad \Rightarrow\Rightarrow \qquad P_{BC} = -\,163.5 \text{ kN} \qquad \text{(n)}$$

The minus signs in these results indicate that both members BC and BF are subjected to internal compression forces.

Step 5: Solve for the allowable design stresses in the members.

Let's determine the allowable design stresses from Eq. (4.2) as:

$$\sigma_{Design} = S_y/\mathbf{SF_y} = (290)/(3.25) = 89.23 \text{ MPa} \qquad \text{(o)}$$

Step 6: Determine the areas required to limit the design stresses to the allowable level determined in Eq. (n).

$$A = P/\sigma_{Design} = P/(89.23) \qquad \text{(p)}$$

The results from Eq. (o) are presented in Table 5.2.

Table 5.2
Results showing required dimensions for scissors truss members

Member	P (kN)	A (mm^2)	D_1 (mm)	D_2 (mm)
AB	–188.7	2,115	51.89	55
AF	167.7	1,879	48.91	50
BC	–163.5	1,832	48.30	50
BF	–22.33	250.3	17.85	20

Inspection of Table 5.2 indicates the areas for each member. We have assumed that round rods with a diameter D are to be employed in fabricating the scissors truss. In Table 5.2, the diameter D_1 is the computed value. Because rods of these sizes are not commercially available, we have increased the diameter to D_2 that corresponds to standard sizes that are available without special order.

5.3 ZERO-FORCE MEMBERS

Zero-force members are usually easy to identify in a structure, if we examine all of the joints that do not carry externally applied loads. Let's consider three examples of joints with a zero-force member as shown in Fig. 5.7.

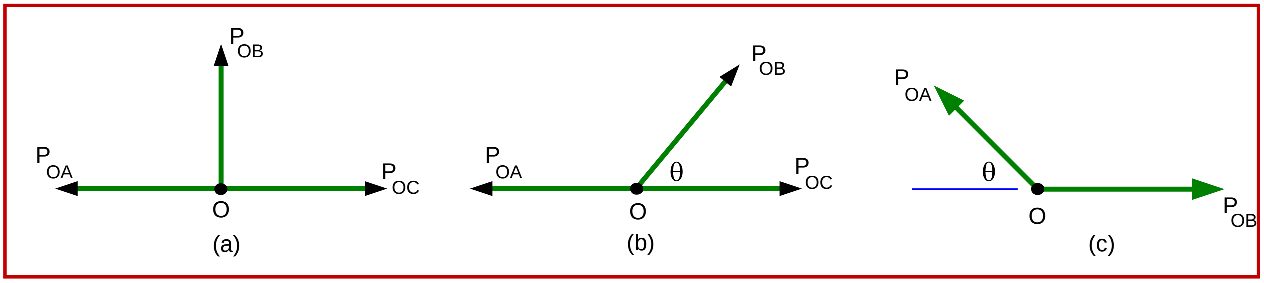


Fig. 5.7 Joint O with a zero-force member OB.

Let's begin by considering the equilibrium of joint O in Fig. 5.7a. Write $\Sigma F_y = 0$ and it is clear that:

$$\sum F_y = P_{OB} = 0 \qquad \text{(a)}$$

because the forces P_{OA} and P_{OC} do not have force components in the y direction. Also, from $\Sigma F_x = 0$, we may write that:

$$\sum F_x = -P_{OA} + P_{OC} = 0$$

$$P_{OA} = P_{OC} \qquad \text{(b)}$$

Next, examine joint O in Fig. 5.7b. While force P_{OB} is inclined at some angle θ with respect to the x-axis, its component in the y direction is zero because neither P_{OA} nor P_{OC} have components in the y direction. Because P_{OB} is transmitted through a two-force member, the component of P_{OB} in the x direction is also zero. Thus, OB is a zero-force member.

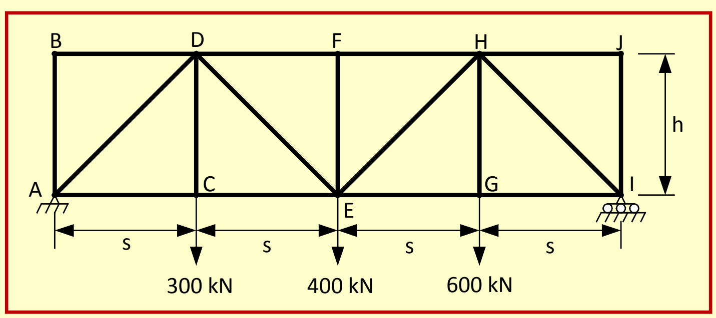
Finally, consider the joint in Fig. 5.7c. Summing forces in the y direction yields $P_{OA} = 0$. Then summing forces in the x direction gives $P_{OB} = 0$. In this case, both forces at joint O are zero. The three cases illustrated in Fig. 5.7 provide the only three configurations where zero force members can be identified by inspection.

EXAMPLE 5.4

For the truss defined in Fig. E5.4, determine the forces, stresses, and safety of members GI, HI, JI and HJ. The dimensions h = s = 6 m. The yield strength of the structural steel from which the truss is fabricated is 285 MPa. The following cross sectional areas have been specified for these members:

$$A_{GI} = 8{,}000\ \text{mm}^2,\ A_{HI} = 10{,}000\ \text{mm}^2,\ A_{JI} = 1{,}600\ \text{mm}^2,\ \text{and } A_{HJ} = 1{,}800\ \text{mm}^2$$

Fig. E5.4



Solution:

Step 1: Draw a FBD enabling you to write equations for the unknown reaction forces, illustrated in Fig. E5.4a.

Step 2: Use the equilibrium relations to solve for the reaction forces R_{Ax}, R_{Ay} and R_{Iy}.

$$\Sigma M_A = (4s)R_{Iy} - (3s)(600) - (2s)(400) - (s)(300) = 0$$

$$R_{Iy} = (1/4)[1{,}800 + 800 + 300] = 725.0 \text{ kN} \qquad \text{(a)}$$

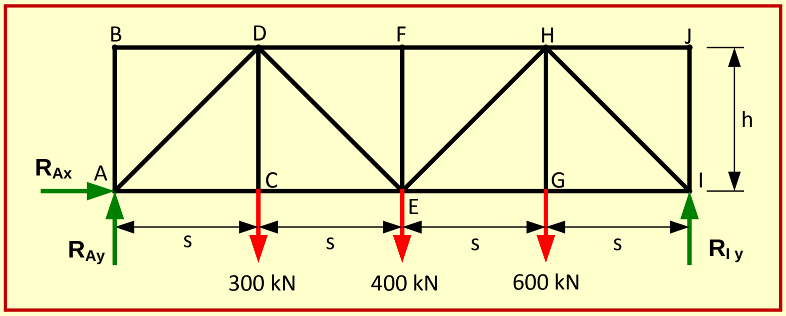


Fig. E5.4a

$$\Sigma F_y = R_{Ay} + R_{Iy} - 300 - 400 - 600 = 0$$

$$R_{Ay} = 1{,}300 - 725.0 = 575.0 \text{ kN} \qquad \text{(b)}$$

$$\Sigma F_x = R_{Ax} = 0$$

Step 3: Select joints J and I and construct FBDs of each showing the internal and external forces acting at these pins, as shown in Fig. 5.4b.

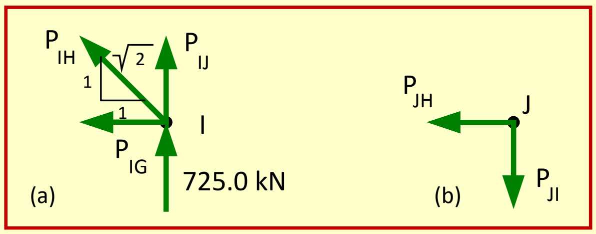


Fig. E5.4b

Step 4: Use equilibrium relations as necessary to solve for the unknown forces.

Consider joint J.

It is evident by inspection of the FBD for the joint at point J that:

$$P_{JH} = P_{JI} = 0 \qquad \text{(c)}$$

Both JI and JH are zero-force members that do not contribute to the strength or rigidity of the structure.

For joint I, we write:

$$\Sigma F_y = 725.0 + [(1)/(\sqrt{2})]P_{IH} = 0$$

$$P_{IH} = -(\sqrt{2})(725.0) = -1{,}025.3 \text{ kN} \qquad \text{(d)}$$

$$\Sigma F_x = -P_{IG} - [(1)/(\sqrt{2})]P_{IH} = 0$$

$$P_{IG} = -[(1)/(\sqrt{2})]P_{IH} = -[(1)/(\sqrt{2})](-1025.3) = 725.0 \text{ kN} \qquad \text{(e)}$$

The analysis of the forces in the four members defined in the problem statement is complete.

Step 5: Determine the stresses in member IH and IG. Note, the stresses in the zero-force members are obviously zero.

$$\sigma_{IH} = P_{IH} / A_{IH} = -(1{,}025.3 \times 10^3)/(10{,}000) = -102.5 \text{ N/mm}^2 = -102.5 \text{ MPa} \qquad \text{(f)}$$

$$\sigma_{IG} = P_{IG} / A_{IG} = (725.0 \times 10^3)/(8{,}000) = 90.63 \text{ N/mm}^2 = 90.63 \text{ MPa} \qquad \text{(g)}$$

Step 6: Compute the safety factor and the margin of safety for these two members.

For member IH:

$$SF = S_y/\sigma = (285)/(102.5) = 2.780$$

$$MOS = SF - 1 = 2.780 - 1 = 1.780 = 178.0\% \qquad \text{(h)}$$

For member IG

$$SF = S_y/\sigma = (285)/(90.63) = 3.145$$

$$MOS = SF - 1 = 3.145 - 1 = 2.145 = 214.5\% \qquad \text{(i)}$$

An examination of the results indicates that the zero-force members contribute nothing to the strength and rigidity of the structure. They should be removed to reduce the weight and cost of the structure unless there is another compelling reason for their inclusion in the design. The other two members of the truss (IG and IH) appear to be properly sized with safety factors of about 3. One member IH is loaded in compression and its resistance to buckling must be ascertained in a subsequent analysis.

5.4 METHOD OF SECTIONS

The method of sections, like the method of joints is based on construction of a FBD and the subsequent application of the equations of equilibrium. The difference is in the selection of the portion of the truss used for the FBD. With the method of sections, we cut the truss and remove a complete section of it for the FBD. A typical example of a section cut through a truss and the two FBDs that are produced is shown in Fig. 5.8. The section cut passes through three members of the truss—BD, CD and CE. You have the option of examining either the left or right hand side of the truss. FBDs of both sides are shown in Fig. 5.8. At the location of the section cut, the internal forces in the members are displayed in the FBDs.

The FBDs provide a guide in writing the appropriate equilibrium relations. For the sections of the truss, either the right hand side or its left, the system of forces is classified as planar, but not concurrent. Accordingly, the following three equilibrium equations apply:

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0; \qquad \text{and} \qquad \Sigma M_O = 0$$

Next solve for three unknown internal forces in members of the truss exposed by the section cut. Consider an example to demonstrate the procedure for the method of sections, and follow the same six-step process described for the method of joints.

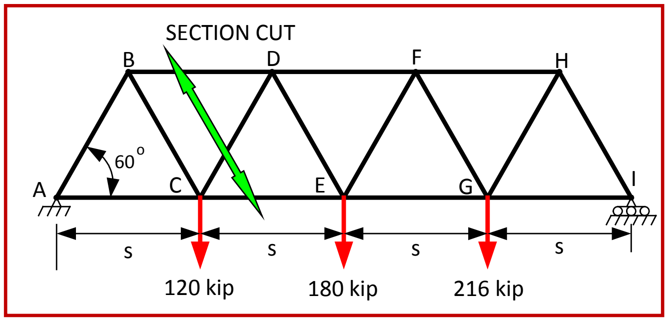
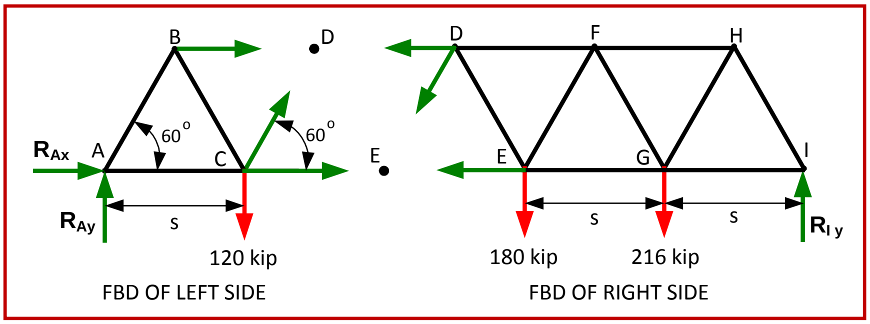


Fig. 5.8 FBDs resulting from the section cut of the truss.



EXAMPLE 5.5

Consider the truss made from equilateral triangles, shown in Fig. 5.8, and determine the margin of safety for members BD, CD and CE. The span s of each triangular section is 25 ft and the yield strength of the structural steel used in fabrication is 41 ksi. The cross sectional areas of the three members are A_{BD} = 14 in^2, A_{CD} = 9 in^2 and A_{CE} = 22 in^2.

Solution:

Step 1: Construct a FBD of the entire truss structure, as shown in Fig. E5.5.

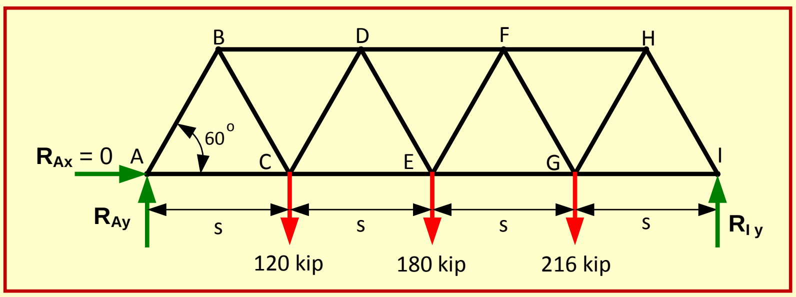


Fig. E5.5

Step 2: Apply the equilibrium relations to determine the reactions at the supports.

$$\Sigma F_x = R_{Ax} = 0$$

$$\Sigma M_A = (4s)R_{Iy} - (3s)(216) - (2s)(180) - (s)(120) = 0$$

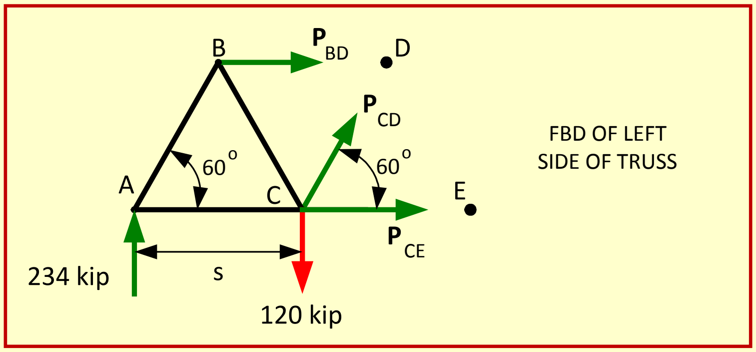
$$R_{Iy} = (¼)[648 + 360 + 120] = 282 \text{ kip} \qquad (a)$$

$$\Sigma F_y = R_{Ay} - 120 - 180 - 216 + 282 = 0$$

$$R_{Ay} = 234 \text{ kip} \qquad (b)$$

Step 3: Select a portion of the truss and make a section cut through the three members BD, CD and CE. Construct a FBD of that portion, as shown in Fig. E5.5a. In this example, the left side is used for the FBD, because it is easier to analyze than the right side.

Fig E5.5a



Step 4: Apply the equilibrium relations to solve for the forces P_{BD}, P_{CD} and P_{CE}.

$$\Sigma M_C = -(234)(s) - [P_{BD}\sin(60)](s) = 0$$

$$P_{BD} = -(234)/(0.8660) = -270.2 \text{ kip} \quad \text{(c)}$$

$$\Sigma F_y = 234 - 120 + P_{CD}\sin(60) = 0$$

$$P_{CD} = (234 - 120)/(0.8660) = -131.6 \text{ kip} \quad \text{(d)}$$

$$\Sigma F_x = P_{BD} + P_{CD}\cos(60) + P_{CE} = 0$$

$$P_{CE} = -(-270.2) - (-131.6)(0.5) = 336.0 \text{ kip} \quad \text{(e)}$$

Step 5: Determine the stresses in the members.

$$\sigma_{BD} = P_{BD}/A_{BD} = -(270.2)/(14) = -19.30 \text{ ksi} \quad \text{(f)}$$

$$\sigma_{CD} = P_{CD}/A_{CD} = -(131.6)/(9) = -14.62 \text{ ksi} \quad \text{(g)}$$

$$\sigma_{CE} = P_{CE}/A_{CE} = (336.0)/(22) = 15.27 \text{ ksi} \quad \text{(h)}$$

Step 6: Determine the safety factors and margins of safety for the three members.

For member BD:

$$SF = S_y/\sigma = (41)/(19.30) = 2.124; \quad MOS = SF - 1 = 2.124 - 1 = 112.4\% \quad \text{(i)}$$

For member CD:

$$SF = S_y/\sigma = (41)/(14.62) = 2.804; \quad MOS = SF - 1 = 2.804 - 1 = 180.4\% \quad \text{(j)}$$

For member CE:

$$SF = S_y/\sigma = (41)/(15.27) = 2.685; \quad MOS = SF - 1 = 2.685 - 1 = 168.5\% \quad \text{(k)}$$

From an examination of the results for these three truss members, it is apparent that the design is reasonable with safety factors exceeding two in all cases. The design is nearly balanced because the margin of safety is nearly the same for all three members. However, members BD and CD are loaded in compression and their critical buckling load must be determined to complete the analysis.

EXAMPLE 5.6

For the truss shown in Fig. E5.6, determine the cross sectional area required for members DF, EF and EG if the safety factor for the design is to exceed 3.8. The span s for each section of the truss is 9 m. The yield strength of the hot rolled steel members used in fabricating the truss is 300 MPa.

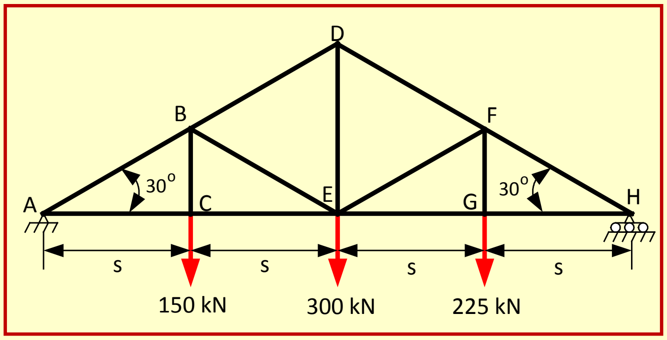


Fig. E5.6

Solution:

Step 1: We construct a FBD of the entire structure as shown in Fig. E5.6a.

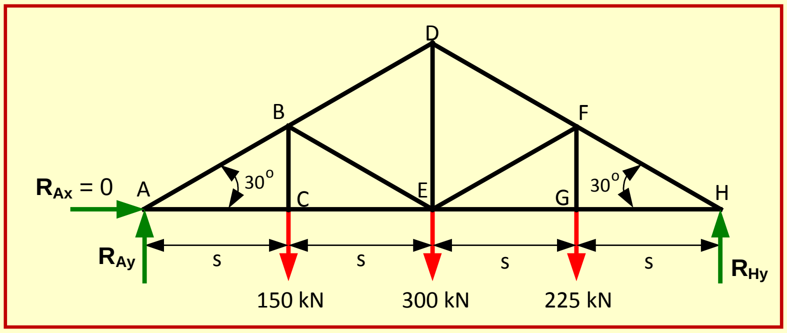


Fig E 5.6a

Step 2: Apply the equilibrium relations to solve for the reaction forces R_{Ay} and R_{Hy}.

$$\Sigma F_x = R_{Ax} = 0$$

$$\Sigma M_A = -(150)(s) - (300)(2s) - (225)(3s) + (4s)R_{Hy} = 0$$

$$R_{Hy} = (1/4)[150 + 600 + 675] = 356.25 \text{ kN} \qquad \text{(a)}$$

$$\Sigma F_y = R_{Ay} - 150 - 300 - 225 + R_{Hy} = 0$$

$$R_{Ay} = 675 - 356.25 = 318.75 \text{ kN} \qquad \text{(b)}$$

Step 3: Make a section cut through the truss to expose the internal forces in members DF, EF and EG. Construct a FBD of the right hand side of the truss showing the unknown internal forces P_{DF}, P_{EF}, and P_{EG}, as presented in Fig. E5.6b.

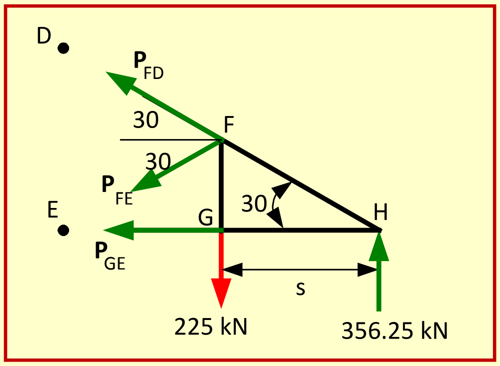


Fig. E5.6b

Step 4: Apply the equilibrium relations to solve for the unknown internal forces.

$$\Sigma M_F = (356.25)(s) - (P_{GE})[(s)\tan(30)] = 0$$

$$P_{GE} = (356.25)/[\tan(30)] = 617.0\text{ kN} \qquad \text{(c)}$$

$$\Sigma F_y = 356.25 - 225 - P_{FE}\sin(30) + P_{FD}\sin(30) = 0$$

$$P_{FE} = P_{FD} + 262.5 \qquad \text{(d)}$$

$$\Sigma F_x = -P_{GE} - P_{FD}\cos(30) - P_{FE}\cos(30) = 0$$

$$P_{FD} = -[P_{FE} + P_{GE}/\cos(30)] \qquad \text{(e)}$$

Substituting Eqs. (c) and (d) into Eq. (e) yields:

$$P_{FD} = -[P_{FE} + (617.0)/(0.8660)] = -[(P_{FD} + 262.5) + 712.5]$$

$$P_{FD} = -487.5\text{ kN} \qquad \text{(f)}$$

Next, substitute Eq. (f) into Eq. (d) to obtain:

$$P_{FE} = P_{FD} + 262.5 = (-487.5) + 262.5 = -225.0\text{ kN} \qquad \text{(g)}$$

Step 5: Determine the stresses in the three members.

$$\sigma_{GE} = P_{GE}/A_{GE}$$

$$\sigma_{FD} = P_{FD}/A_{FD} \qquad \text{(h)}$$

$$\sigma_{FE} = P_{FE}/A_{FE}$$

It is impossible to solve for the stresses implicitly, because the cross sectional areas of the three members are unknown. We will use Eqs. (h) after developing a relation for the stresses in terms of the safety factor. The result will give us a relationship between the allowable stress and the internal force on each member.

Step 6: Determine the allowable stresses and then the cross sectional areas by using the information regarding the safety factor and the yield strength required for the truss members.

$$SF = S_y/\sigma \qquad \sigma = S_y/SF \qquad \text{(i)}$$

Substitute Eq. (i) into Eqs. (h) and solve for the cross sectional area of each member.

$$\sigma_{GE} = P_{GE}/A_{GE} = S_y/SF = (300)/(3.8) = 78.95\text{ MPa}$$

$$\sigma_{FD} = P_{FD}/A_{FD} = S_y/SF = (300)/(3.8) = 78.95\text{ MPa} \qquad \text{(j)}$$

$$\sigma_{FE} = P_{FE}/A_{FE} = S_y/SF = (300)/(3.8) = 78.95\text{ MPa}$$

From Eqs. (j), (c), (f) and (g), it is evident that:

$$A_{GE} = P_{GE}/\sigma_{GE} = (617.0 \times 10^3)/(78.95) = 7{,}815 \text{ mm}^2$$

$$A_{FD} = P_{FD}/\sigma_{FD} = (487.5 \times 10^3)/(78.95) = 6{,}175 \text{ mm}^2 \qquad \text{(k)}$$

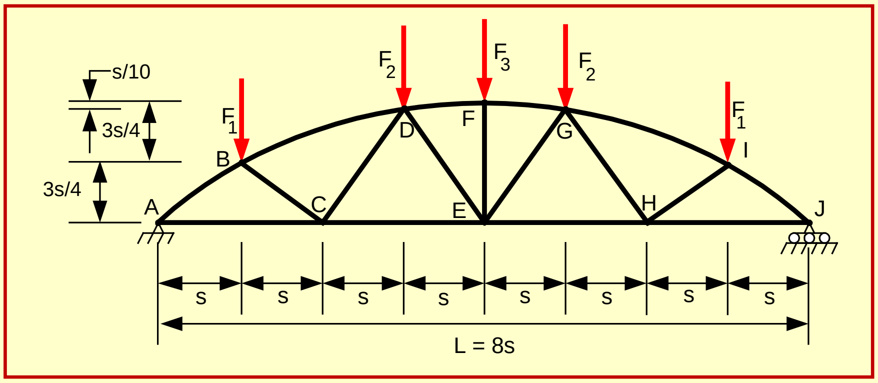
$$A_{FE} = P_{FE}/\sigma_{FE} = (225.0 \times 10^3)/(78.95) = 2{,}850 \text{ mm}^2$$

The results of Eq. (k) give the cross sectional areas required to achieve the specified safety factors. However, sizes such as these are not normally used in design. Instead, standard size sections slightly larger than those determined in the analysis are sought. Hot rolled steel sections are available in a wide range of sizes and shapes from a large number of suppliers of metal products. Standard sizes are employed, because this practice reduces costs and delivery time.

EXAMPLE 5.7

Determine the forces and the stresses in members CE, DE and DF, of the bowstring truss shown in Fig. E5.7. Note that the numerical parameters that control the forces and stresses are F_1 = 18.0 kip, F_2 = 15.0 kip, F_3 = 12.0 kip, s = 10 ft, A_{CE} = 3.0 in.^{2}, A_{DE} = 2.0 in.^{2}, and A_{DF} = 4.0 in.^{2}

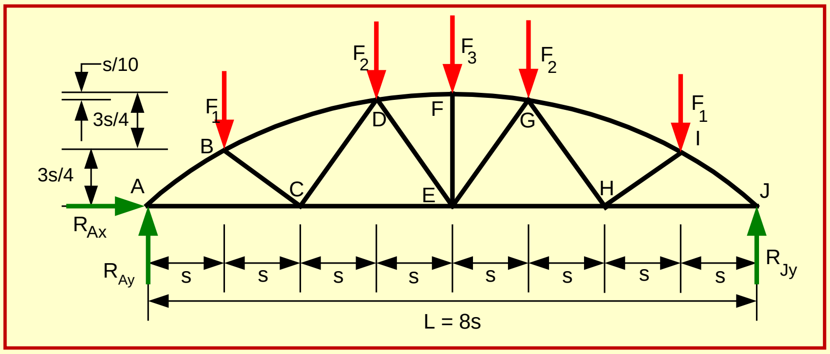
Fig. E5.7



Solution:

Step 1: We construct a FBD of the entire structure as shown in Fig. E5.7a.

Fig. E5.7a



Step 2: Apply the equilibrium relations to solve for the reaction forces R_{Ax}, R_{Ay} and R_{Jy}.

$$\Sigma F_x = R_{Ax} = 0 \qquad \text{(a)}$$

Because the loading and the geometry of the truss are symmetric, we may write:

$$R_{Ay} = R_{Jy} \qquad \text{(b)}$$

$$\Sigma F_y = R_{Ay} - 2F_1 - 2F_2 - F_3 + R_{Jy} = 0$$

$$R_{Ay} = R_{Jy} = (1/2)[(2)(18.0) + (2)(15.0) + 12.0] = 39.0 \text{ kip} \qquad \text{(c)}$$

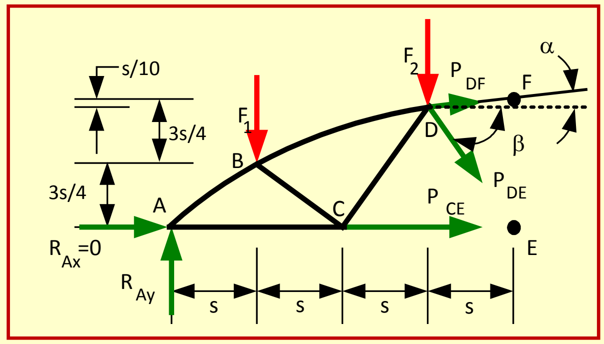
Step 3: Make a section cut through the truss to expose the internal forces in members CE, DE, and DF. Construct a FBD of the left hand side of the truss showing the unknown internal forces P_{CE}, P_{DE}, and P_{DF} as presented in Fig. E5.7b. All members in a truss are two force members, which implies that the direction of the forces coincide with the axis of the bars. For curved two force members, the action line of the forces is along a straight line drawn between the two end points of the bar as shown in Fig. E5.7c.

Fig. E5.7b

$\alpha = \tan^{-1}(1/10) = 5.71°$

$\beta = \tan^{-1}[(6/4) - (1/10)]$

$\beta = \tan^{-1}(1.4) = 54.46°$



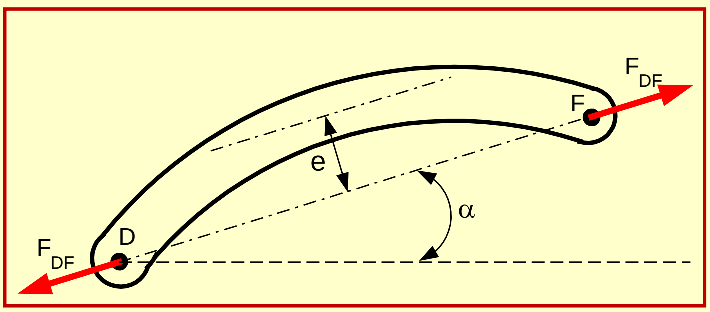


Fig. E5.7c The action line of the two forces is along a straight line drawn between the two end points of the bar

Step 4: Apply the equilibrium relations to solve for the unknown internal forces. Select point E and write $\Sigma M_E = 0$ to obtain:

$$\Sigma M_E = (s)F_2 + (3s)F_1 - (4s)R_{Ay} - (P_{DF})[(s)\sin(\alpha) + (1.4s)\cos(\alpha)] = 0 \qquad \text{(d)}$$

Substituting numerical parameters into Eq. (d) and solving for P_{DF} yields:

$$P_{DF} = -58.29 \text{ kip} \qquad \text{(e)}$$

Consider equilibrium of the sectioned structure in the y direction:

$$\Sigma F_y = R_{Ay} - F_1 - F_2 + P_{DF}\sin(\alpha) - P_{DE}\sin(\beta) = 0 \qquad \text{(f)}$$

Substituting numerical parameters into Eq. (f) and solving for P_{DE} yields:

$$P_{DE} = 0.2464 \text{ kip} \qquad \text{(g)}$$

Consider equilibrium of the sectioned structure in the x direction:

$$\Sigma F_x = R_{Ax} + P_{CE} + P_{DF} \cos(\alpha) + P_{DE} \cos(\beta) = 0 \qquad \text{(h)}$$

Substituting numerical parameters into Eq. (h) and solving for P_{CE} yields:

$$P_{CE} = 57.86 \text{ kip} \qquad \text{(i)}$$

The minus sign for the numerical result for member DF indicates that it is subjected to a compressive force. However, the other two members of the bowstring truss are in tension.

Step 5: Determine the stresses in the three members. We may solve for the stresses in members DE and CE using Eq. (3.4) as:

$$\begin{aligned} \sigma_{DE} &= P_{DE}/A_{DE} = (0.2464 \text{ kip})/(2.0 \text{ in.}^2) = 0.1232 \text{ ksi} \\ \sigma_{CE} &= P_{CE}/A_{CE} = (57.86 \text{ kip})/(3.0 \text{ in.}^2) = 19.29 \text{ ksi} \end{aligned} \qquad \text{(j)}$$

It is not possible for us to determine the stresses in member DF because it is curved. The curvature of the member does not affect the determination of the force P_{DF}, because DF is a two-force member. For this reason, the internal forces act along a line of action passing through points D and F on the bowstring truss, as shown in Fig. E5.7c.

The external force F_{DF} is equal to the internal force P_{DF}, both of which act along the line of action defined with respect to the x-axis by the angle α. However, the centerline of the curved member does not coincide with the line of action of the two-forces. Indeed, an eccentricity e is evident in Fig. E5.7c. This eccentricity is important when determining the stresses in the curved member, because it causes bending of the member. The relation for stresses given by Eq. (3.4) is not valid in this case, because it does not account for either the curvature of the member or the eccentricity of the load line. Both of these topics are beyond the scope of this textbook, and will be covered later in a more advanced course in Mechanics of Materials.

5.5 SUMMARY

Trusses are fabricated from bars and cables to provide structures capable of spanning long distances. As such, they are often used in the design of bridges and roofs to cover large buildings. Stability of the truss is achieved by constructing it from a series of triangular arrangements.

The bars or cables in a truss are two force members connected together by means of joints. We assume that the joints are capable of rotation and cannot support moments. Because the bars and cables are long and flexible, it is necessary to restrict the point of load application to only the trusses' joints.

Two methods for analysis of trusses are described. The first, the method of joints, entails removing a specified joint from the truss and constructing a FBD showing both the internal and external forces acting on it. The joint is represented as a particle acted upon by a planar, concurrent force system. As such two equilibrium relations apply:

$$\Sigma F_x = 0 \qquad \text{and} \qquad \Sigma F_y = 0$$

These two relations are sufficient to solve for any two unknown forces acting on the joint.

The method of sections involves making a section cut through the truss to divide it into two or more parts. A FBD is prepared for a section of the truss showing both internal and external forces. The forces acting on a particular section are planar, but they are not concurrent; hence, the equilibrium relations that apply are:

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0; \qquad \text{and} \qquad \Sigma M_O = 0$$

These relations are sufficient to solve for any three unknown forces that act on the portion of the truss under consideration. Examples illustrating the procedure for using the method of joints and the method of sections were provided.

A procedure was described for identifying zero-force members in a truss by examining only those joints not subjected to externally applied loads.

1. When a joint is acted upon by three internal forces and two of these forces are collinear, then the remaining force must be zero.
2. When two non-collinear members meet at a joint, where no external load is applied, both are zero force members.

Zero force members do not contribute to the strength or rigidity of a structure and should be eliminated unless there is a compelling reason for their inclusion in the design of the truss.

The margin of safety (MOS) is often used to characterize the safety of trusses and is related to the safety factor (SF) by the equation:

$$MOS = SF - 1 \qquad (5.2)$$

Typical values of MOS range from 100-300%, which correspond to safety factors ranging from 2-4.

CHAPTER 6

PROPERTIES OF AREAS

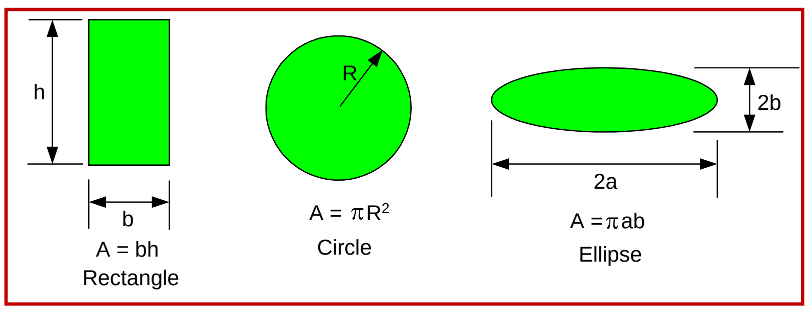
6.1 <u>AREA</u>

The cross sectional area, A of structural members plays an extremely important role in the efficiency and the adequacy of any member to safely carry its load. For example, the stress in a uniaxial tension or compression member is given by:

$$\sigma = P/A \qquad (3.4)$$

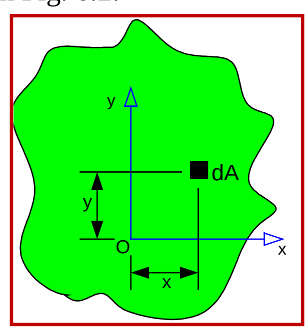
The force P is in the numerator and the area A of the cross section is in the denominator of this relation for the normal stresses. To lower the stresses we have only two options — decrease the force P or increase the area A. The area A of the cross section depends on its shape as indicated in Fig. 6.1:

Fig. 6.1 Dimensions and equations for areas of common cross-sections.



For areas of arbitrary shape, we determine their area by integration as indicated in Fig. 6.2:

Fig. 6.2 An arbitrary area A.



The area, A is determined by summing the incremental area dA in an integration process:

$$A = \int_A dA \qquad (6.1)$$

If the boundaries of the area are not known in terms of well-defined mathematical expressions, integrating to determine the area A is not possible. However, it is always possible to divide the area into many small squares or rectangles each with an area ΔA. If these squares or rectangles closely follow the boundary of the shape in question and completely fill the interior region, the area is then given by:

$$A = \sum_{i=1}^{N} \Delta A_i = N\Delta A \tag{6.2}$$

where N is the number of small squares or rectangles.

6.2 FIRST MOMENT OF AN AREA

The first moment of an area is important, because it is useful in locating the position of the centroid of a given cross sectional area. Let's consider an arbitrary area with the coordinate system Oxy, as shown previously in Fig. 6.2. The first moment of the area A about the x-axis is defined as:

$$Q_x = \int_A y\,dA \tag{6.3}$$

Also, the first moment of the area A with respect to the y-axis is defined as:

$$Q_y = \int_A x\,dA \tag{6.4}$$

Depending on the location of the coordinate system relative to the area, the numerical values obtained for the first moments Q_x and Q_y may be either positive or negative. The units for Q_x and Q_y are mm^3 in the SI system or in^3 in the U. S. Customary system.

6.3 CENTROID OF AN AREA

The centroid of an area A is defined by point C located relative to an arbitrary coordinate system Oxy, as illustrated in Fig. 6.3. The centroid is defined as the point that locates the center of gravity of a line, an area or a volume.

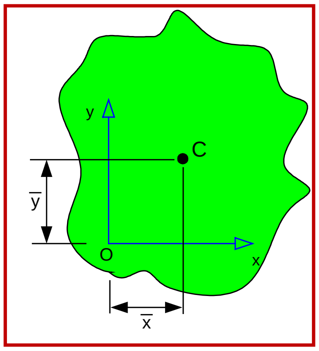


Fig. 6.3 The centroid C of an area A is located with coordinates $\bar{x}$ and $\bar{y}$.

The coordinates $\bar{x}$ and $\bar{y}$ locating the centroid of an area are determined from the first moments as:

$$Q_y = \int_A x\,dA = A\bar{x}$$

$$Q_x = \int_A y\,dA = A\bar{y} \tag{6.5}$$

where $\bar{x}$ and $\bar{y}$ are dimensions locating the centroid, as shown in Fig. 6.3.

Let's illustrate the method for determining the first moment of the area and the location of a centroid by considering a few elementary shapes, in the examples presented below.

EXAMPLE 6.1

Consider the rectangular area illustrated in Fig E6.1, with the origin of an Oxy coordinate system positioned at its lower left-hand corner. Determine the first moments of the rectangular area and the location of its centroid relative to this coordinate system.

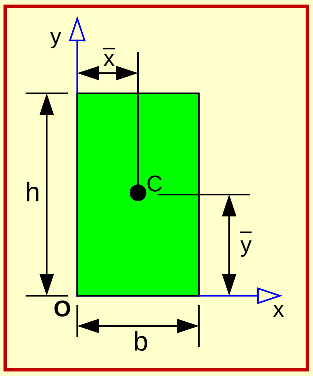


Fig E6.1 A rectangle with a coordinate system located along its edges.

Solution:

For the rectangle area presented in Fig. E6.1, the first moments of the area about the x and y axes are given by Eq. (6.5) as:

$$Q_x = A\bar{y} = (bh)\frac{h}{2} = \frac{bh^2}{2}$$

$$Q_y = A\bar{x} = (bh)\frac{b}{2} = \frac{b^2h}{2} \qquad (6.6)$$

It was possible to quickly solve for the first moments, Q_x and Q_y, because we recognized the location of the centroid for the rectangular area. When an axis of symmetry exists for a given area, the centroid is located somewhere on this axis of symmetry. With the rectangle, two axes of symmetry exist; hence, the location of the centroid is at the intersection of its two symmetric axes.

For the circular cross section shown in Fig. 6.4, the center of the circle locates the centroid. The center also serves as the origin C for a special set of axes known as the centroidal axes x_c and y_c.

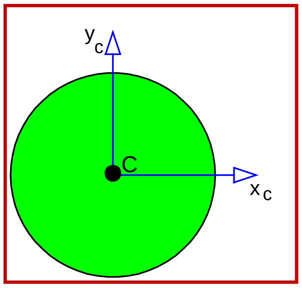


Fig. 6.4 The centroid serves as the origin for the centroidal axes x_c, y_c.

For cross sectional shapes such as ellipses, circles, squares and rectangles, the center may be located by inspection, because these geometries have two axes of symmetry. However, for non-symmetric figures, such as triangles, portions of circles, parabolic areas, etc., locating the center of the area is not obvious. In Example 6.2 we demonstrate a method for determining the centroid's location for an area that does not exhibit two axes of symmetry.

With respect to a centroidal coordinate system, the first moment of the area must vanish for both axes. Therefore:

$$Q_{\bar{x}} = \int_A y dA = 0 \qquad Q_{\bar{y}} = \int_A x dA = 0 \qquad (6.7)$$

These relations are employed to locate the centroid of an area of any shape providing its boundary can be defined with some mathematical function.

EXAMPLE 6.2

For a right triangle, determine the first moment of the area about its base and vertical side, and the position of its centroid relative to these two sides. The right triangle with a base b and a height h is illustrated in Fig. E6.2:

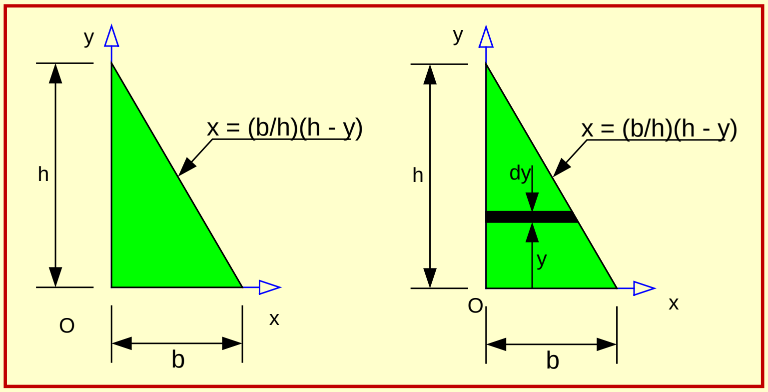


Fig E6.2 A right triangle with a coordinate system coincident with its base and vertical side.

Solution:

To begin, let's determine the first moment of the area of the right triangle relative to the x-axis (its base side). Writing Eq. (6.3) gives:

$$Q_x = \int_A y dA = \int_y y(x dy) \qquad (a)$$

where dA = x dy is located a distance y from the x axis.

Note, the equation for the inclined boundary of the triangle is given by:

$$x = (b/h)(h - y) \qquad (b)$$

The limits on the integral go from 0 to h to encompass the area of the triangle. We substitute Eq. (b) and the limits on y for the integral into Eq. (a), and write:

$$Q_x = \frac{b}{h}\int_0^h (h - y) y dy \qquad (c)$$

Integrating Eq. (c) gives:

$$Q_x = \frac{b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h \qquad (d)$$

Evaluating Eq. (d) gives:

$$Q_x = bh^2/6 \qquad (6.8)$$

By using Eq. (6.4) and following the same procedure, we find the first moment about the vertical side of the triangle is given by:

$$Q_y = b^2h/6 \qquad (6.9)$$

Equation (6.8) gives the first moment of the area of a right triangle about its base. This is an interesting exercise in calculus, but what does it have to do with determining the location of the centroid of the right triangle? The results presented in Eqs. (6.8) and (6.9) are intermediate steps. We continue the solution by combining the results of Eqs. (6.8) and (6.9) with Eqs. (6.5) to obtain:

$$Q_x = bh^2/6 = A\,\bar{y} = (bh/2)\,\bar{y} \qquad (e)$$

$$Q_y = b^2h/6 = A\,\bar{x} = (bh/2)\,\bar{x} \qquad (f)$$

where $\bar{x}$ and $\bar{y}$ locate the Cx_cy_c coordinates relative to the Oxy coordinates (see Fig. E6.2a).

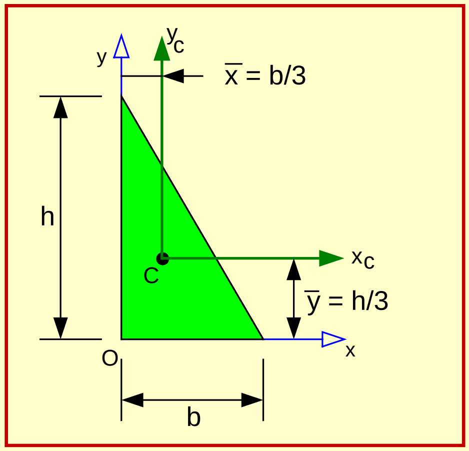


Fig. E6.2a A right triangular area with a base axes Oxy and centroidal axes Cx_cy_c.

To determine the position of the centroid, let's solve Eqs. (e) and (f) for $\bar{x}$ and $\bar{y}$ to obtain:

$$\bar{y} = h/3 \qquad \bar{x} = b/3 \qquad (6.10)$$

We employ Eqs. (6.3) and (6.4) to determine the first moment of the area Q relative to either the x or y axes. The location of the centroid is then established from Eq. (6.5). The location of the centroid for common shapes is well known. These results are presented together with a drawing of the shape of the area in Fig. 6.5.

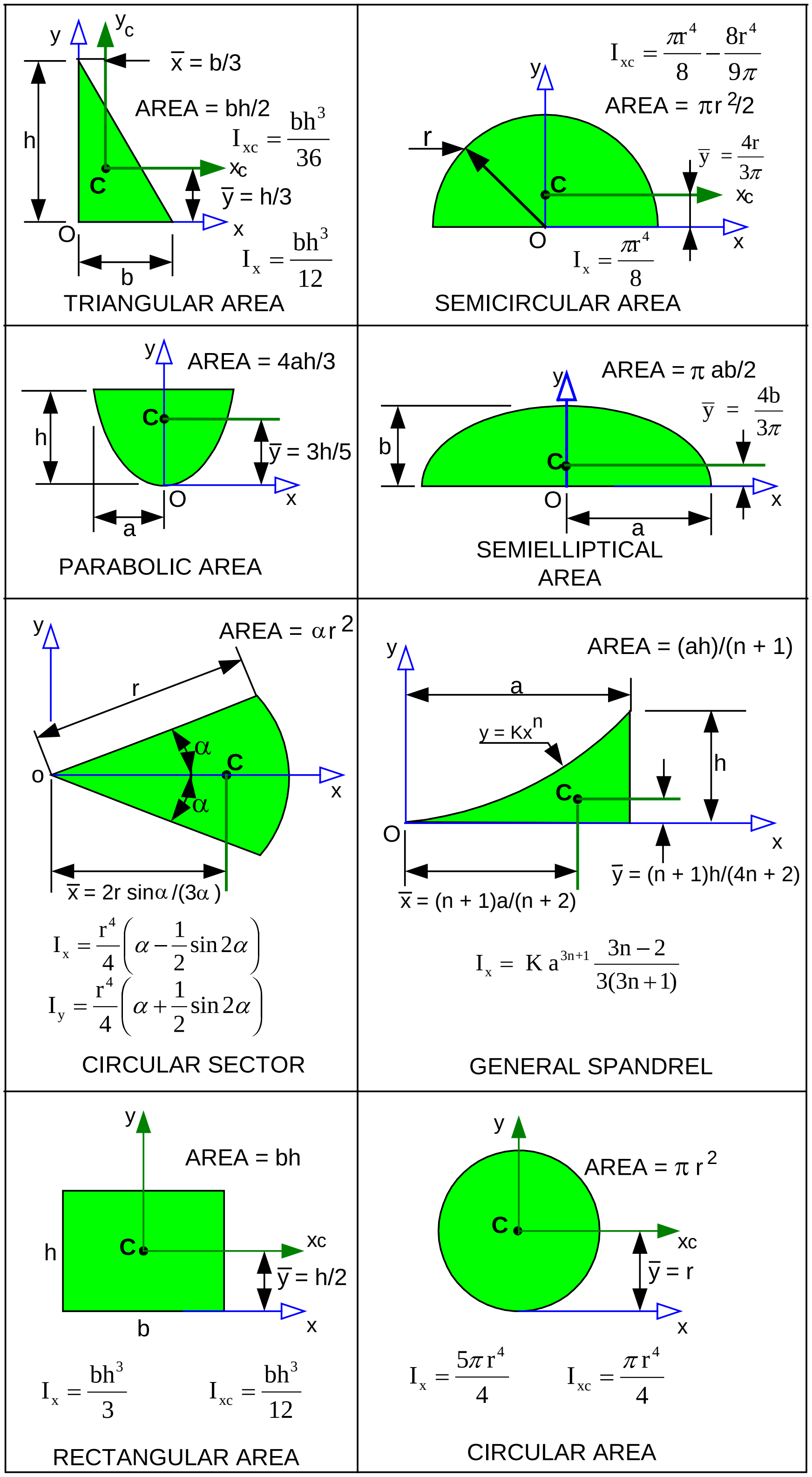
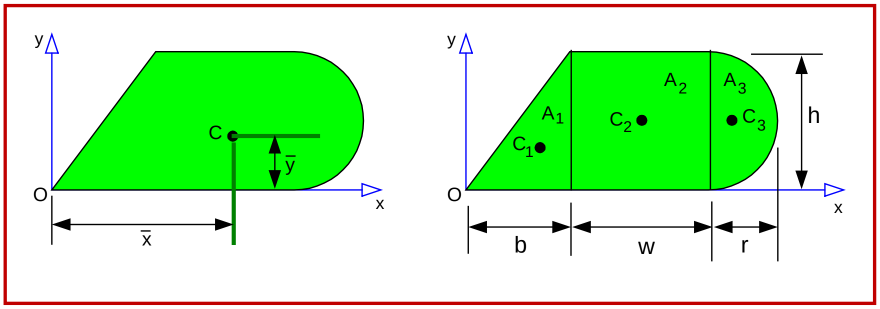


Fig. 6.5 Area properties of some common shapes.

6.4 LOCATING THE CENTROID OF A COMPOSITE AREA

In many cases, the shape of a cross section is unusual and differs from the common geometries described in Fig. 6.5. To locate the centroid of areas with irregular shapes, we divide its area into several different common shapes for which solutions for the location of the centroid are known. Then we combine the product of these individual areas and their centroid locations to give the location of the centroid of the composite area. Let's consider the irregular shape defined in Fig. 6.6, and locate the position of its centroid by employing the composite area technique.

Fig. 6.6 The unusual shaped area on the left is divided into common shaped areas on the right.



EXAMPLE 6.3

Determine the location of the centroid of the irregular shape defined in Fig. 6.6.

Solution:

To begin the solution, we divide the area into three sub-regions—A_1, A_2 and A_3 as shown in Fig. 6.6. Note the sub-regions are a right triangle, rectangle and semicircle. The dimensions of the three different shapes are:

- The triangle — base b = 12 units and height h = 18 units.
- The rectangle — width w = 24 units and height h = 18 units.
- The semicircle — radius r = 9 units.

We apply Eq. (6.5) to the composite area, and write:

$$\Sigma Q_x \quad \Rightarrow \quad \overline{Y}\ A_t = \Sigma\ \overline{y}_n\ A_n \qquad (6.11)$$

$$\Sigma Q_y \quad \Rightarrow \quad \overline{X}\ A_t = \Sigma\ \overline{x}_n\ A_n \qquad (6.12)$$

where $\overline{X}$ and $\overline{Y}$ are the coordinates of the centroid of the irregular (composite) area, and A_t is the total area of the composite area. Let's first determine $\overline{Y}$ from Eq. (6.11):

$$\overline{Y}\,(A_1 + A_2 + A_3) = \overline{y}_1\, A_1 + \overline{y}_2\, A_2 + \overline{y}_3\, A_3 \qquad \text{(a)}$$

Solving for $\overline{Y}$ yields:

$$\overline{Y} = (\overline{y}_1\, A_1 + \overline{y}_2\, A_2 + \overline{y}_3\, A_3)/(A_1 + A_2 + A_3) \qquad \text{(b)}$$

Substituting results from Fig. 6.5 into Eq. (b) gives:

$$\bar{Y} = \frac{\left(\frac{h}{3}\left(\frac{bh}{2}\right) + \frac{h}{2}(wh) + r\left(\frac{\pi r^2}{2}\right)\right)}{\left(\frac{bh}{2} + wh + \frac{\pi r^2}{2}\right)} \qquad \text{(c)}$$

Substituting b = 12, h = 18, w = 24 and r = 9 units into Eq. (c) yields:

$$\bar{Y} = 8.514 \text{ units}$$

This result is slightly less than h/2 as we would anticipate. The presence of the triangle shifts the location of the centroid downward from the centerline of the rectangle. Next, let's determine the position of the centroid in the direction of the x-axis. We begin by using Eq. (6.12), and write:

$$\bar{X}\,(A_1 + A_2 + A_3) = \bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3 \qquad \text{(d)}$$

Solving for $\bar{X}$ yields:

$$\bar{X} = (\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3)/(A_1 + A_2 + A_3) \qquad \text{(e)}$$

From the information listed in Fig. 6.5, we determine the centroid location for each of the shapes in the composite area as indicated below:

$$\bar{X} = \frac{\left(\frac{2b}{3}\left(\frac{bh}{2}\right) + \left(b + \frac{w}{2}\right)wh + \left(b + w + \frac{4r}{3\pi}\right)\frac{\pi r^2}{2}\right)}{\left(\frac{bh}{2} + wh + \frac{\pi r^2}{2}\right)} \qquad \text{(f)}$$

Substituting b = 12, h = 18, w = 24 and r = 9 units into Eq. (f) yields:

$$\bar{X} = 24.43 \text{ units} \qquad \text{(g)}$$

We note that the location of the centroid of the irregular area is slightly to the right of the center of the rectangular area. This position is to be expected because the orientation of the right triangle with its area concentrated toward the right side tends to shift the centroid to the right.

6.5 SECOND MOMENT OF THE AREA

The second moment of the area is also known as the area moment of inertia. You will encounter the second moment of the area in your study of Mechanics of Materials, when determining the stresses produced in beams by an internal moment, and in this course when determining the load required to buckle long slender columns. Three different second moments of the area A, illustrated in Figs. 6.7 and 6.8, are defined in accordance to the axes referenced.

The second moment of the area is referenced to one or both of its coordinate axes. The moment of inertia relative to the y and z-axes shown in Fig. 6.7 is defined by:

$$I_z = \int_A y^2 dA \qquad I_y = \int_A z^2 dA \qquad (6.13)$$

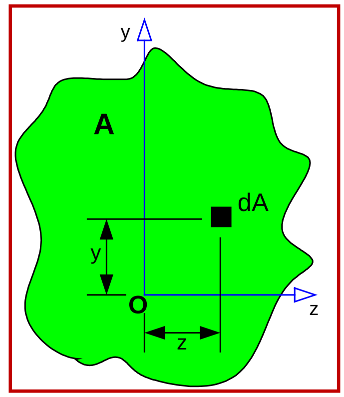


Fig. 6.7 An elemental area dA is used when integrating to determine the second moment of the area A.

We also define a polar moment of inertia of the area A relative to the origin O of the y – z coordinate system as indicated in Fig. 6.8 as:

$$J_O = \int_A r^2 dA = \int_A (z^2 + y^2) dA = \int_A z^2 dA + \int_A y^2 dA \qquad (6.14)$$

and from Eq. (6.13) it is evident that:

$$J_0 = I_y + I_z \qquad (6.15)$$

The units for the second moments of the area are in^4 in the U. S. Customary system and m^4 or mm^4 in the SI system.

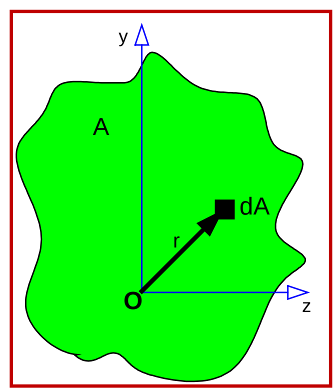


Fig. 6.8 Coordinate system for the polar moment of inertia J_O.

Finally, the radius of gyration of an area A with respect to its axes is defined by:

$$I_z = r_z^2 A \qquad I_y = r_y^2 A \qquad J_O = r_O^2 A \qquad (6.16)$$

The symbols r_z and r_y reference the radii of gyration relative to the z and y-axes, respectively. The radius of gyration for the polar moment of inertia is r_O. The radius of gyration is simply a number when squared and multiplied by the area give the area moment of inertia.

Let's consider a few examples to demonstrate the use of Eq. (6.13) to determine the second moment of the area.

EXAMPLE 6.4

For the rectangular cross sectional area and coordinate system defined in Fig E6.4, determine the equations for I_z and I_y.

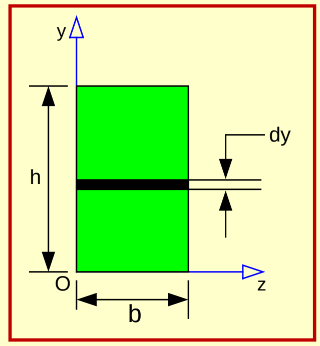


Fig. E6.4 A rectangular area with a coordinate system along its edges.

Solution:

To determine I_z, we write Eq. (6.13) and observe that dA = b dy. Then:

$$I_z = \int_A y^2 dA = b\int_0^h y^2 dy = b\left[\frac{y^3}{3}\right]_0^h = \frac{bh^3}{3} \tag{6.17}$$

Similarly for I_y, we write Eq. (6.13) and observe that dA = h dz. Then:

$$I_y = \int_A z^2 dA = h\int_0^b z^2 dz = h\left[\frac{z^3}{3}\right]_0^b = \frac{hb^3}{3} \tag{6.18}$$

To show the importance of the location of the coordinate system in determining the second moment of the area, shift the origin of the coordinate system to the center of the rectangle, as shown in Fig 6.9.

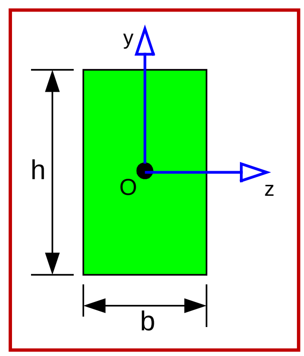


Fig. 6.9 A rectangular area with a centroidal coordinate system.

EXAMPLE 6.5

Determine the moment of inertia of a rectangular area relative to its centroidal axes.

Solution:

To determine the moment of inertia I_z, we write Eq. (6.13) and note that dA = b dy.

$$I_z = \int_A y^2 dA = b\int_{-h/2}^{h/2} y^2 dy = b\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = \frac{bh^3}{12} \tag{6.19}$$

To determine the moment of inertia I_y, we write Eq. (6.13) and note that dA = h dz.

$$I_y = \int_A z^2 dA = h\int_{-b/2}^{b/2} z^2 dz = h\left[\frac{z^3}{3}\right]_{-b/2}^{b/2} = \frac{hb^3}{12} \qquad (6.20)$$

A comparison of the results for the moments of inertia for the rectangle clearly indicates the importance of the location of the coordinate system relative to the area in question. Because we may be required to determine the moment of inertia about axes with arbitrary locations, a useful method for accounting for shifting the position of axes is presented in Section 6.6.

6.6 THE PARALLEL AXIS THEOREM

Let's again consider an arbitrary area A positioned some distance from the z-axis, as shown in Fig 6.10. Assume that the centroidal axis of the area A is known and identified with the z' axis. Also assume that the second moment of the area with respect to the centroidal axis z' is known. Let's compute the moment of inertia I_z with respect to an axis z that is parallel to the centroidal axis, but located some distance d below it. We begin again with Eq. (6.13) and write:

$$I_z = \int_A y^2 dA = \int_A (y_1 + d)^2 dA = \int_A y_1^2 dA + 2d\int_A y_1 dA + d^2\int_A dA \qquad (6.21)$$

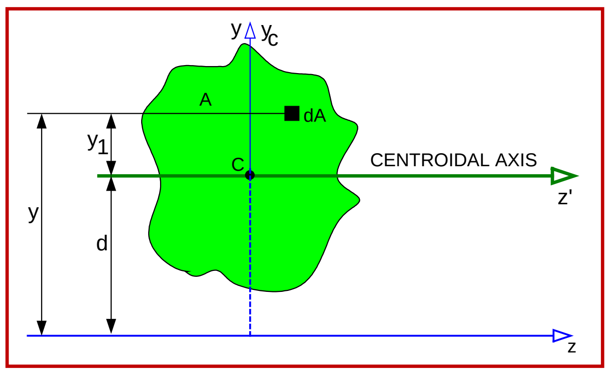


Fig. 6.10 An arbitrary area A positioned a distance d from the z-axis.

The first term on the right hand side of Eq. (6.21) is the moment inertia $I_{z'}$ of the area about the centroidal axis. The second term $\int y_1 dA$ is the first moment of the area about its centroidal axis, which is zero by definition of the centroid. The final term is simply Ad^2. Hence, the parallel axis theorem may be written as:

$$I_z = I_{z'} + Ad^2 \qquad (6.22)$$

where $I_{z'}$ is the second moment of the area about the centroidal axis.

EXAMPLE 6.6

To demonstrate the use of Eq. (6.22), let's determine the moment of inertia I_z of the rectangle shown in Fig E6.6.

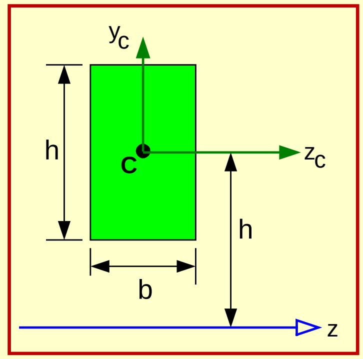


Fig. E6.6 A rectangular area shifted by an amount d = h relative to the z-axis.

Solution:

To determine the moment of inertia I_z, we recall Eq. (6.22) and write:

$$I_z = I_{z'} + Ad^2 \qquad \text{(a)}$$

From Eq. (6.19) it is clear that $I_{z'} = bh^3/12$. Then from Eq. (6.22) we obtain:

$$I_z = bh^3/12 + (bh)(h^2) = (13/12)\, bh^3 \qquad \text{(b)}$$

This example illustrates three points. First, the parallel axis theorem is very helpful in determining the increase in the moment of inertia when the reference axis is some distance removed from the centroidal axis. Second, the moment of inertia is very sensitive to the movement of the reference axis relative to the centroidal axis. In this example, we moved the reference axis by an amount equal to the height h of the section and increased the inertia by a factor of 12. Third, the moment of inertia is a minimum about the centroidal axis.

6.7 MOMENTS OF INERTIA OF COMPOSITE AREAS

To determine the moment of inertia of areas with complex shapes, we divide the complex area into subsections. Each subsection is a simple shape, such as a square, rectangle, circle or semi-circle with known properties. To demonstrate the procedure for determining its moment of inertia, consider a structural tee with a web and a flange, as illustrated in Fig. 6.11.

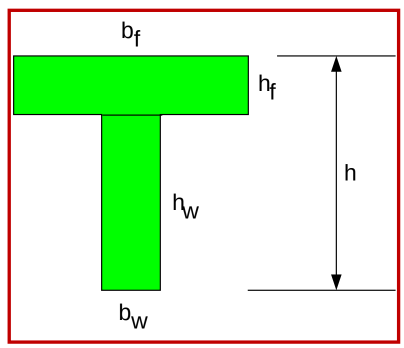


Fig. 6.11 A structural tee is divided into two rectangular areas A_{web} and A_{flange}.

The procedure for determining the properties of the composite area representing the structural tee involves three steps:

- Determine the location of the centroidal axis of the composite area.
- Determine the moment of inertia of each area of the composite section about its centroidal axis.

- Employ the parallel axis theorem to determine the moment of inertia of the total section relative to its centroidal axis.

EXAMPLE 6.7

Determine the moment of inertia I_z of the structural tee, shown in Fig. E6.7, relative to its centroidal axis. The dimensions of the structural tee are given by:

$$h = 300 \text{ mm},\ h_f = 45 \text{ mm},\ h_w = 255 \text{ mm},\ b_f = 210 \text{ mm, and } b_w = 31.5 \text{ mm.}$$

Solution:

To determine the location of the centroidal axis of the composite area, subdivide the structural tee into two rectangular areas A_1 and A_2, as shown in Fig. E6.7. Then apply Eq. (6.11) to determine the location of the centroid relative to the z (reference) axis.

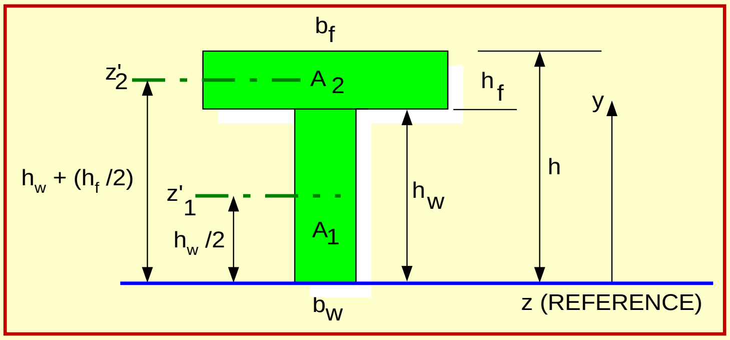


Fig. E6.7 Subdivide the structural tee to form two rectangular areas and establish a convenient reference axis z.

From Eq. (6.11), we write:

$$\Sigma Q_{z(REFERENCE)} = \overline{Y} A_t = \Sigma\, \overline{y}_n A_n = \overline{y}_1 A_1 + \overline{y}_2 A_2 \qquad (a)$$

Substituting the dimensional quantities from Fig. E6.7 into Eq. (a) yields:

$$\overline{Y} = \frac{\left(\frac{h_w}{2}\right)A_1 + \left(h_w + \frac{h_f}{2}\right)A_2}{A_1 + A_2} = \frac{\frac{b_w h_w^2}{2} + b_f h_f h_w + \frac{b_f h_f^2}{2}}{b_w h_w + b_f h_f} \qquad (b)$$

Substitute the given dimensions into Eq. (b) to obtain:

$$\overline{Y} = 208.6 \text{ mm} \qquad (c)$$

Next determine the moment for inertia of each area of the composite section about its own centroidal axis. For A_1 (the web) with a width b_w = 31.5 mm and a height h_w = 255 mm, we use Eq. (6.19) and write:

$$I_{z'1} = b_w h_w^3/12 = (31.5)(255)^3/12 = 43.53 \times 10^6 \text{ mm}^4 \qquad (d)$$

For A_2 (the flange) with a width b_f = 210 mm and a height h_f = 45.0 mm, we write:

$$I_{z'2} = b_f h_f^3/12 = (210)(45)^3/12 = 1.595 \times 10^6 \text{ mm}^4 \qquad (e)$$

Finally, employ the parallel axis theorem to determine the moment of inertia of the total area relative to the centroidal axis. Note for the composite area, we express the moment of inertia due to the two areas as:

$$I_z = I_{z1} + I_{z2} \qquad \text{(f)}$$

We use Eq. (6.22) to expand Eq. (f) as:

$$I_z = (I_{z1'} + A_1\, d_1^2) + (I_{z2'} + A_2\, d_2^2) \qquad \text{(g)}$$

The dimensions d_1 and d_2 are given by:

$$d_1 = \overline{Y} - h_w/2 = 208.6 - (255/2) = 81.10 \text{ mm} \qquad \text{(h)}$$

$$d_2 = [h_w + (h_f/2)] - \overline{Y} = [255 + (45/2)] - 208.6 = 68.90 \text{ mm} \qquad \text{(i)}$$

Substituting numerical values for the terms in Eq. (g) yields the final result:

$$I_z = [(43.53\times10^6) + (31.5)(255)(81.1)^2] + [(1.595\times10^6) + (210)(45)(68.9)^2] = 142.8\times10^6 \text{ mm}^4$$

The determination of the moment of inertia for complex shapes is simple but tedious. Care must be exercised to avoid numerical errors in computing each of the quantities shown in Eq. (g). The moment of inertia is important in determining the bending stresses in beams, because I_z occurs in the denominator of the well know flexural formula $\sigma = -My/I_z$. The procedure for establishing the position of the centroidal axis is also important because it locates the neutral axis about which bending occurs. When the position of the neutral axis is known, we can establish y_{max}, I_z and the maximum bending stress σ. The minimum moment of inertia is also used to determine the buckling load for long slender columns, which will be discussed in Chapter 7.

EXAMPLE 6.8

Determine the properties of the unsymmetrical wide flanged section, shown in Fig. E6.8 that are listed below:

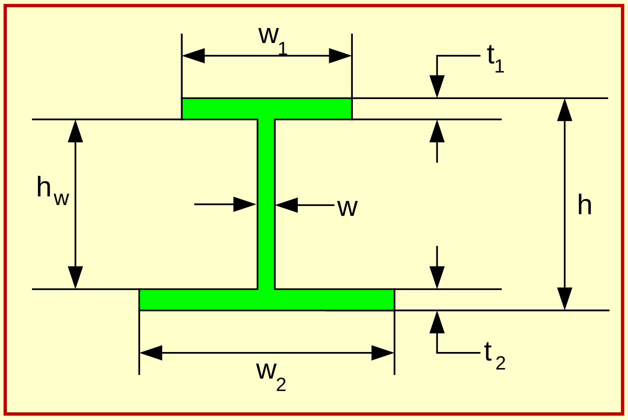


Fig. E6.8 Dimensions of the unsymmetrical wide flanged section.

1. The location of the centroid relative to a defined reference axis.
2. The moment of inertia of the web, top flange and bottom flange relative to the centroidal axis.
3. The moment of inertia of the complete section relative to the centroidal axis.

Solution:

To determine the location of the centroidal axis of the composite area, divide the unsymmetrical wide flanged section into three different areas consisting of the top flange, web and bottom flange, as shown in Fig. E6.8a. Also establish a reference axis, z that is used as the datum for dimensioning the location of the centroid.

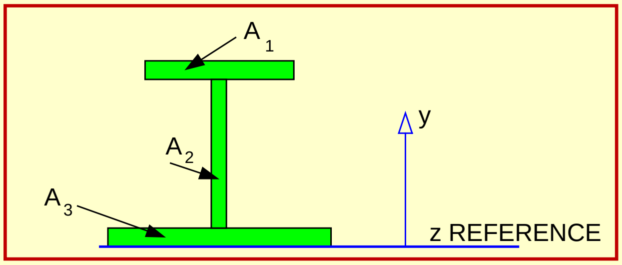


Fig. E6.8a Divide the unsymmetrical wide flanged section into three parts.

Let's apply Eq. (6.11) to determine the location of the centroid of the unsymmetrical wide-flange section relative to the z (reference) axis.

$$\Sigma Q_{z(\text{REFERENCE})} = \bar{Y} A_t = \Sigma \bar{y}_n A_n = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3 \qquad \text{(a)}$$

Substituting the symbols for the dimensions from Fig. E6.8 into Eq. (a) yields:

$$\bar{Y} = \frac{A_1[h_w + t_2 + (t_1/2)] + A_2[t_2 + (h_w/2)] + A_3(t_2/2)}{A_1 + A_2 + A_3}$$

(b)

$$\bar{Y} = \frac{1}{2}\left[\frac{(t_1 w_1)(2h_w + 2t_2 + t_1) + (h_w w)(2t_2 + h_w) + (w_2 t_2)t_2}{w_1 t_1 + h_w w + w_2 t_2}\right]$$

Suppose we assume the proportions of the cross section as:

$$w_1 = 10\text{ w};\ w_2 = 15\text{ w};\ t_1 = 0.125\text{ h};\ t_2 = 0.125\text{ h};\ \text{and } h_w = 0.75\text{ h} \qquad \text{(c)}$$

Then Eq. (b) reduces to:

$$\bar{Y} = 0.4294\text{ h} \qquad \text{(d)}$$

Finally if the height h = 480 mm and web thickness w = 24 mm, the location of the centroidal axis is given by:

$$\bar{Y} = 206.1\text{ mm} \qquad \text{(e)}$$

To determine the moment of inertia of each area of the composite section about its centroidal axis, apply Eq. (6.19) to each of the three rectangular areas to obtain:

$$I_{z'1} = (1/12)w_1 t_1^3 = (1/12)(240)(60)^3 = 4.320 \times 10^6\text{ mm}^4 \qquad \text{(f)}$$

$$I_{z'2} = (1/12)w h_w^3 = (1/12)(24)(360)^3 = 93.31 \times 10^6\text{ mm}^4 \qquad \text{(g)}$$

$$I_{z'3} = (1/12)\, w_2 t_2^3 = (1/12)(360)(60)^3 = 6.480 \times 10^6\text{ mm}^4 \qquad \text{(h)}$$

Next use the parallel axis theorem [Eq. (6.22)] to determine the moment of inertia of each of the three areas relative to the centroidal axis. For the top flange (Area 1):

$$I_{z1} = I_{z'1} + A_1 d_1^2 = I_{z'1} + (w_1\ t_1)[(h - t_1/2) - \overline{Y}\,]^2$$

$$I_{z1} = 4.320 \times 10^6 + (240)(60)[(480 - 30) - 206.1]^2 \qquad \text{(i)}$$

$$I_{z1} = 4.320 \times 10^6 + 856.6 \times 10^6 = 860.9 \times 10^6 \text{ mm}^4$$

For the web (Area 2):

$$I_{z2} = I_{z'2} + A_2 d_2^2 = I_{z'2} + (w h_w)[(t_2 + h_w/2) - \overline{Y}\,]^2$$

$$I_{z2} = 93.31 \times 10^6 + (24)(360)[(60 + 180) - 206.1]^2 \qquad \text{(j)}$$

$$I_{z2} = 93.31 \times 10^6 + 9.929 \times 10^6 = 103.2 \times 10^6 \text{ mm}^4$$

For the bottom flange (Area 3):

$$I_{z3} = I_{z'3} + A_3 d_3^2 = I_{z'3} + (w_2\ t_2)(\overline{Y} - t_2/2)^2$$

$$I_{z3} = 6.480 \times 10^6 + (360)(60)(206.1 - 30)^2 \qquad \text{(k)}$$

$$I_{z3} = 6.480 \times 10^6 + 669.8 \times 10^6 = 676.3 \times 10^6 \text{ mm}^4$$

Before we complete the computation, examine the results listed in Eqs. (i – k). For both of the flanges, we note that the contribution of the $I_{z'}$ term was negligible compared to the Ad^2 term. For the web the $I_{z'}$ term was dominant and the $A_2 d_2^2$ term was negligible. To complete the solution for the moment of inertia of this composite area, we sum the moments of inertia due to the three areas as:

$$I_z = I_{z1} + I_{z2} + I_{z3} \qquad \text{(l)}$$

Substituting the results for Eqs. (i – k) into Eq. (l) yields:

$$I_z = (860.9 + 103.2 + 676.3) \times 10^6 = 1{,}640.4 \times 10^6 \text{ mm}^4 \qquad \text{(m)}$$

The contribution of the web to the moment of inertia about the centroidal axis of this unsymmetrical wide flange section was a small part of the total (6.3%). This example shows that the main purpose of the web is to move the flanges outward from the neutral (centroidal) axis. The height of the web is limited by buckling, where it becomes elastically unstable.

6.8 SUMMARY

Properties of areas is an important topic in mechanics, because stresses produced in structures and machine components depend upon the cross sectional area of the member, its centroidal location and the first and second moments of the area. Most of the equations required to compute the quantities found in the study of Statics, Mechanics of Materials and Dynamics are introduced in this chapter. Many of the topics described here should be familiar to the reader from prior studies of Calculus and Physics.

The area of a shape that can be defined by a mathematical function is determined by integration according to:

$$A = \int_A dA \qquad (6.1)$$

If the shape cannot be expressed with a mathematical function, numerical methods are used to sum small areas ΔA to obtain the total area A.

$$A = \sum_{i=1}^{N} \Delta A_i = N\Delta A \qquad (6.2)$$

The first moment of the area about the x or y axes is defined as:

$$Q_x = \int_A y dA \qquad (6.3)$$

$$Q_y = \int_A x dA \qquad (6.4)$$

The first moment of the area provides a method for determining the centroid locations as:

$$Q_y = \int_A x dA = A\bar{x}$$

$$Q_x = \int_A y dA = A\bar{y} \qquad (6.5)$$

The first moment of the area with respect to the centroidal axes vanishes, thus enabling a technique for determining the location of the centroid of an arbitrary area.

$$Q_{\bar{x}} = \int_A y dA = 0 \qquad Q_{\bar{y}} = \int_A x dA = 0 \qquad (6.7)$$

For complex shapes, the location of the centroid can be determined by dividing the complex shape into a composite of common shapes and summing first moments as indicated below:

$$\Sigma Q_x \quad \Rightarrow \quad \bar{Y} A_t = \Sigma\, \bar{y}_n A_n \qquad (6.11)$$

$$\Sigma Q_y \quad \Rightarrow \quad \bar{X} A_t = \Sigma\, \bar{x}_n A_n \qquad (6.12)$$

The second moment of the area also called the moment of inertia is determined from:

$$I_z = \int_A y^2 dA \qquad I_y = \int_A z^2 dA \qquad (6.13)$$

The polar moment of inertia is determined from:

$$J_O = \int_A r^2 dA = \int_A (z^2 + y^2) dA = \int_A z^2 dA + \int_A y^2 dA \qquad (6.14)$$

and

$$J_0 = I_y + I_z \qquad (6.15)$$

The radius of gyration is related to the area and the moment of inertia by:

$$I_z = r_z^{\,2} A \qquad I_y = r_y^{\,2} A \qquad J_O = r_O^{\,2} A \qquad (6.16)$$

The parallel axis theorem, used to determine the moment of inertia about a parallel axis some distance removed from a centroidal axis, is derived from the following relation:

$$I_z = \int_A y^2 dA = \int_A (y_1 + d)^2 dA = \int_A y_1^2 dA + 2d \int_A y_1 dA + d^2 \int_A dA \qquad (6.21)$$

Equation (6.21) reduces to:

$$I_z = I_{z'} + Ad^2 \qquad (6.22)$$

Moments of inertia for complex shapes can be determined by the following process: (1) divide the body into a composite of simple shapes, (2) calculate the centroid location of the complex shape using Eqs. (6.11) and (6.12), (3) calculate the moment of inertia for each simple shape by using the parallel axis theorem Eq. (6.22), and (4) sum together all the individual moments of inertia.

CHAPTER 7

BUCKLING OF COLUMNS

7.1 INTRODUCTION

In this chapter, the concept of instability of columns under the action of a compressive force is introduced. This concept is new and it leads to a markedly different mode of failure of many structural elements, such as columns, plates and shells. Even the analysis employed differs from our normal approach. In the previous chapters, we were concerned with stresses and deflections. If the stresses were less than the material's strength, the structural element was safe. If the deflections did not exceed some specified limit, the structural element was satisfactory. This new stability concept requires designers to determine a critical buckling load to insure that the structural element will not become unstable and collapse prior to failure by either yield or fracture.

When the compressive force on a column exceeds some critical value P_{CR}, the column becomes unstable and collapses. This mode of failure is extremely dangerous, because the collapse is catastrophic. When a column buckles, without shedding load, it initially deforms in the shape of a half sine wave and then simply folds when a plastic hinge forms at its center, as illustrated in Fig. 7.1.

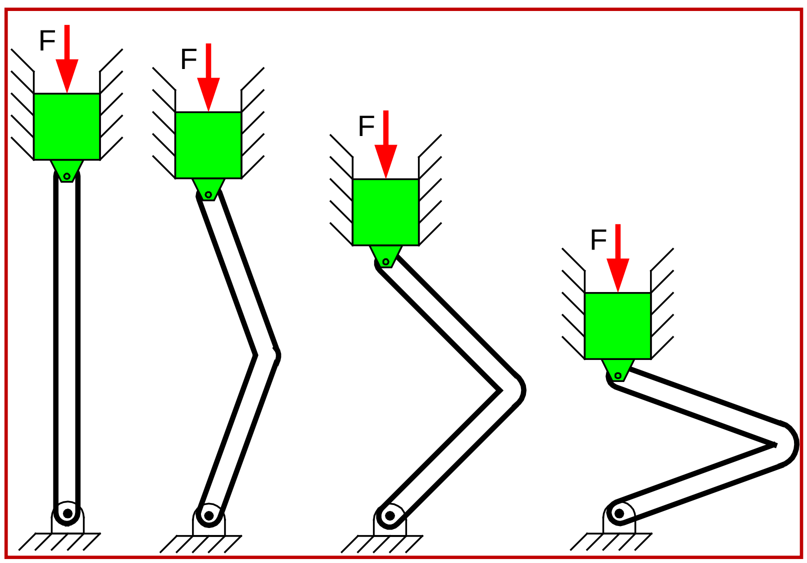


Fig. 7.1 Catastrophic collapse of a column with pinned ends during a buckling event.

In developing the equations for predicting the buckling of a column, we deviate from our previous practice of neglecting the small deformations of a structural element under the action of a load. When computing stress from $\sigma = P/A$, we do not take into account the change in A due to the strains induced by the force P. However, the concept of elastic instability is based on load-induced deformations. In analyzing columns, we consider the moment produced by the axial force P multiplied by the column's transverse deflection y. To show this new analytical approach, consider a column with both ends pinned subjected to a centric force P.

7.2 BUCKLING OF COLUMNS WITH BOTH ENDS PINNED

Consider a column of length L with both ends pinned subjected to a centric force P, as illustrated in Fig. 7.2a. As the magnitude of the centric force increases, the column will bow slightly, as indicated in Fig 7.2b. The column is, at this stage of loading, elastic and stable. To perform the analysis, we model the pinned end supports and the column, as shown in Fig. 7.2c. We have selected the x-axis in the vertical direction and the y-axis in the horizontal direction to adapt the column coordinate system to that used in developing beam theory, which will be introduced in the Mechanics II class. The origin is placed at point A at the top of the column.

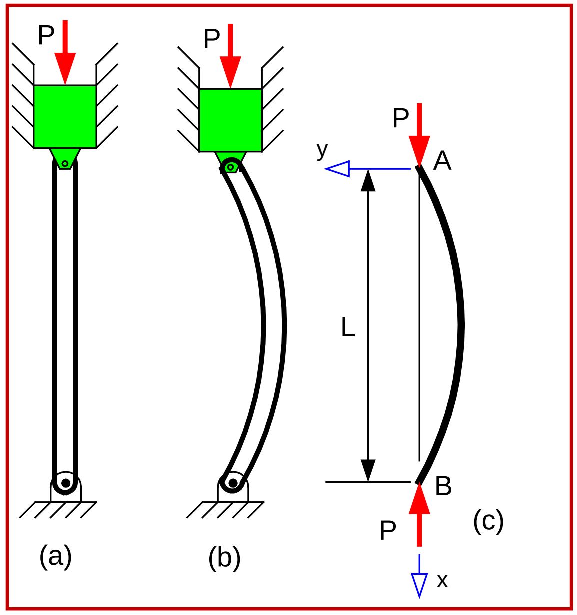


Fig. 7.2 Modeling a column with pinned ends.

Next, make a section cut and remove a portion of the top of the column as a FBD, as shown in Fig. 7.3b.

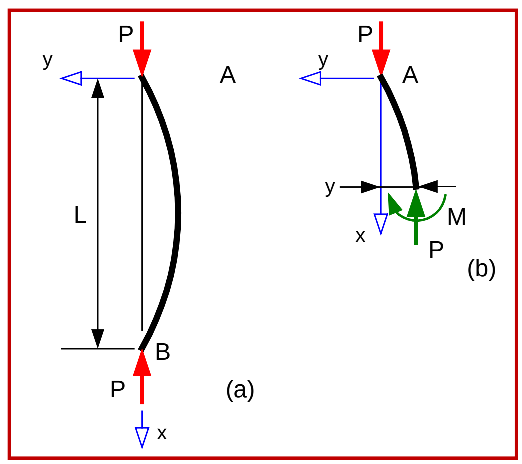


Fig. 7.3 FBD of the column's top portion.

An examination of the axial forces in the FBD shows a couple, which produces a bending moment M = –Py. Considering elementary beam theory from Mechanics II, it can be shown that:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{Py}{EI} \tag{7.1}$$

Rearranging Eq. (7.1) and writing this differential equation in standard format yields:

$$\frac{d^2y}{dx^2} + p^2 y = 0 \tag{7.2}$$

$$\text{where } p = \sqrt{\frac{P}{EI}} \tag{7.3}$$

Equation (7.2) is a linear, second-order differential equation with constant coefficients, which can be solved for y to obtain:

$$y = C_1 \sin(px) + C_2 \cos(px) \qquad (7.4)^{1}$$

where C_1 and C_2 are constants of integration.

We solve for the constants of integration using boundary conditions. Because both ends of the column are pinned, they cannot deflect. We use this boundary condition by writing:

$$y(0) = 0 \qquad\qquad y(L) = 0 \tag{7.5}$$

It is clear from the first of the boundary conditions listed in Eq. (7.5) that $C_2 = 0$. Substituting the second boundary condition ($y = 0$ at $x = L$) into Eq. (7.4) yields:

$$C_1 \sin(pL) = 0 \tag{7.6}$$

There are two interpretations for Eq. (7.6). First, the equation is satisfied if $C_1 = 0$, which gives $y = 0$ for all of x. This is not realistic, because it implies that the column remains straight and will never buckle. Because buckling has been demonstrated in the laboratory and observed in the field, this solution is discarded. The remaining solution to Eq. (7.6) is obtained from values of pL which permit $\sin(pL) = 0$.

$$\text{If} \quad pL = n\pi \qquad \text{then} \qquad \sin(pL) = 0 \tag{a}$$

Then from Eq. (a) and Eq. (7.3), we write:

$$p = \sqrt{\frac{P}{EI}} = \frac{n\pi}{L} \tag{7.7a}$$

and

$$P = \frac{n^2 EI \pi^2}{L^2} \tag{7.7b}$$

It is necessary to interpret Eq. (7.7b) because it contains a **counter n**, which can be 0, 1, 2, etc. We seek the lowest meaningful value of n. Clearly $n = 0$ does not produce a realistic result, because this value gives $P = 0$ indicating the column is unstable prior to loading. We seek the lowest value for $P > 0$; hence we select $n = 1$, to obtain:

$$\boxed{P_{CR} = \frac{\pi^2 EI}{L^2}} \tag{7.8}$$

where P_{CR} is the critical centric compression load that initiates buckling and catastrophic collapse of the column.

[1] This solution can be verified by differentiating Eq. (7.4) twice, with respect to x to obtain d^2y/dx^2. Then substitute d^2y/dx^2 and y into Eq. (7.2).

The moment of inertia I is the minimum value for the cross sectional area of the column. Equation (7.8) is known as the Euler relation for elastic buckling. It is valid for long slender columns[2] with both ends pinned.

Returning to Eq. (7.4), we may use Eq. (7.7a) and express the deflection of the column as:

$$y = C_1 \sin\left(\frac{\pi x}{L}\right) \tag{7.9}$$

This result indicates that the column deflects as a sinusoid, when the load $P = P_{CR}$. The fact that we cannot determine the amplitude C_1 of the sin wave is not important, because the column has become unstable and will quickly collapse. Let's apply the Euler buckling theory to determine the critical load for two different columns both with pinned ends.

EXAMPLE 7.1

A column 20 feet long is fabricated from an aluminum tube with a 6.0 in. outside diameter and a 5.0 in. inside diameter. Determine the critical buckling force if the tube is pinned at both ends.

Solution:

We employ Eq. (7.8) to write the expression for the critical buckling force as:

$$P_{CR} = \frac{\pi^2 EI}{L^2} \tag{a}$$

The modulus of elasticity for an aluminum alloy is $E = 10.4 \times 10^6$ psi = 10,400 ksi, as obtained from Appendix B-1. The minimum value of the moment of inertia for a tube can be determined from Fig. 6.5 as:

$$I = (\pi/4)[r_o^4 - r_i^4] = (\pi/64)[d_o^4 - d_i^4] = (\pi/64)[(6.0)^4 - (5.0)^4] = 32.938 \text{ in.}^4 \tag{b}$$

Substituting Eq. (b) into Eq. (a) yields:

$$P_{CR} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (10,400)(32.938)}{[(20)(12)]^2} = 58.70 \text{ kip} \tag{c}$$

EXAMPLE 7.2

A column 9 m long is fabricated from a square tube (1020 HR steel) shown in Fig. E7.2. Determine the critical buckling force if the tube is pinned at both ends. Also determine the compressive stress in the tube and comment on its magnitude.

[2] We will discuss the distinction among long slender columns, intermediate columns and short columns later in this chapter.

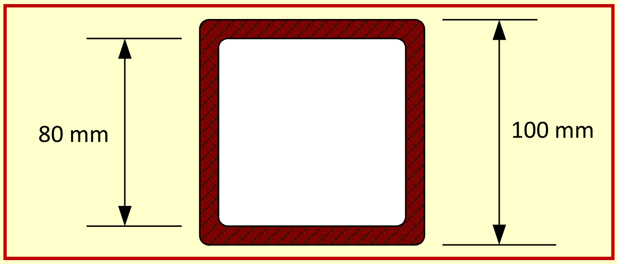


Fig. E7.2

Solution:

We employ Eq. (7.8) to write the expression for the critical buckling force as:

$$P_{CR} = \frac{\pi^2 EI}{L^2} \qquad (a)$$

The modulus of elasticity E = 207 GPa for steel is obtained from Appendix B-1 The minimum value of the moment of inertia for the square tube can be determined from Fig. 6.5 as:

$$I = (1/12)[h_o^4 - h_i^4] = (1/12)[(100)^4 - (80)^4] = 4.920 \times 10^6 \text{ mm}^4 \qquad (b)$$

Recall that 1 GPa = 1 kN/mm^2, and substitute Eq. (b) into Eq. (a) to yield:

$$P_{CR} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (207)(4.920 \times 10^6)}{(9.0 \times 10^3)^2} = 124.1 \text{ kN} \qquad (c)$$

The axial stress produced by the critical force P_{CR} is:

$$\sigma_{CR} = P_{CR}/A = (124{,}100)/[(100)^2 - (80)^2] = 34.47 \text{ MPa} \qquad (d)$$

The yield strength of 1020 HR steel is listed in Appendix B-2 as 290 MPa; hence, the stress at the critical load that produces buckling is only about 12% of the yield strength. This example illustrates the danger of collapse due to buckling when columns are long and slender.

7.3 INFLUENCE OF END CONDITIONS

The critical buckling load is profoundly affected by the end conditions of the column. Increasing the constraint at either end markedly increases the critical load required for the onset of elastic instability. Three additional sets of boundary conditions are considered to show the influence of constraint on buckling load.

1. Top — pinned	Bottom — fixed.
2. Top — free	Bottom — fixed.
3. Top — fixed	Bottom — fixed.

7.3.1 Column Buckling with One Pinned and One Fixed End

Consider the column with its bottom fixed and its top pinned, as shown in Fig. 7.4. It is subjected to an axial load P, as shown in Fig. 7.5. To begin the analysis, draw the deformed shape of the column in its initial stages of instability. In preparing this drawing, recognize that the slope (dy/dx) of the column at the fixed end and the deflection (y) at both its ends must be zero.

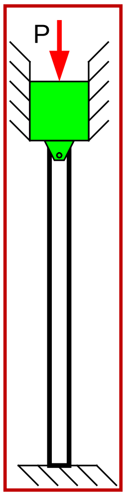


Fig. 7.4 A column with one end pinned and one end fixed.

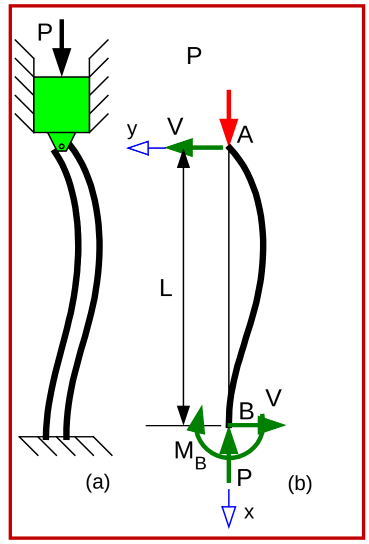


Fig. 7.5 Deformation of the column at initiation of elastic instability.

Next, make a section cut and remove a portion of the top of the column to represent the FBD as shown in Fig. 7.6b.

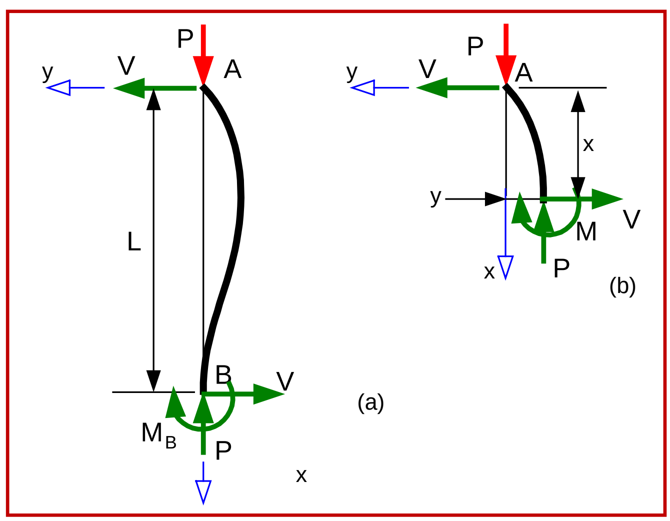


Fig. 7.6 FBD of the top portion of the column.

An examination of the FBD shows two couples produced by the equal and opposite forces V and P. These couples result in a bending moment M given by:

$$M = -Py - Vx \tag{7.10}$$

Substituting Eq. (7.10) into Eq. (7.1) yields:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{1}{EI}\left(Py + Vx\right) \tag{7.11}$$

Rearranging Eq. (7.11) to write the differential equation in standard format gives:

$$\frac{d^2y}{dx^2} + p^2 y = -\frac{Vx}{EI} \tag{7.12}$$

$$\text{recall } p = \sqrt{\frac{P}{EI}}$$

Equation (7.12) is a linear, second-order differential equation with constant coefficients; however, it is not homogeneous. We obtain the homogeneous solution by setting the right hand side of Eq. (7.12) equal to zero and solving for y_h to obtain:

$$y_h = C_1 \sin(px) + C_2 \cos(px) \tag{7.13}$$

where C_1 and C_2 are constants of integration.

The particular solution is an expression for y_p that satisfies Eq. (7.12). Consider the expression for y_p shown below:

$$y_p = -Vx/P \tag{7.14}$$

The expression in Eq. (7.14) satisfies Eq. (7.12); hence, it represents the particular solution. The general solution is given by adding the homogeneous and particular solutions to obtain:

$$y = y_h + y_p = C_1 \sin(px) + C_2 \cos(px) - Vx/P \tag{7.15}$$

We solve for the constants of integration using boundary conditions. Because the top of the column is pinned it cannot deflect; hence, we write:

$$y(0) = 0 \tag{7.16}$$

The boundary conditions for the fixed end reflect the fact that the column cannot deflect and its slope is zero at x = L. Hence we write:

$$y(L) = 0 \qquad \frac{dy}{dx}(L) = 0 \tag{7.17}$$

It is clear from the boundary condition listed in Eq. (7.16) that $C_2 = 0$. Substituting the first boundary condition (y = 0 at x = L) from Eq. (7.17) into Eq. (7.15) yields:

$$C_1 \sin(pL) = VL/P \tag{7.18}$$

Next differentiate Eq. (7.15) with respect to x to obtain:

$$dy/dx = pC_1 \cos(px) - V/P \tag{7.19}$$

Substituting the boundary condition (dy/dx = 0 at x = L) from Eq. (7.17) into Eq. (7.19) gives:

$$pC_1 \cos(pL) = V/P \tag{7.20}$$

Dividing Eq. (7.18) by Eq. (7.20) leads to the transcendental equation:

$$\tan(pL) = pL \qquad (7.21)$$

Solving this transcendental equation by trial and error gives the first meaningful solution for pL as:

$$pL = 4.4934 \qquad (7.22)$$

Then substituting Eq. (7.3) into Eq. (7.22) and solving, we obtain:

$$P_{CR} = \frac{20.19EI}{L^2} \approx \frac{\pi^2 EI}{(0.7L)^2} \qquad (7.23)$$

Comparing the result in Eq. (7.23) with that in Eq. (7.8) shows that the two equations differ only by the coefficient multiplying the term (EI/L^2). Recognizing this similarity, let's write a general equation for the buckling of columns as:

$$P_{CR} = \frac{\pi^2 EI}{(kL)^2} \qquad (7.24)$$

For the column with two pinned ends k = 1, and for the column with one end pinned and the other end fixed k = 0.7. Clearly, fixing one end of the column increased the critical load required for buckling by a factor of slightly more than two.

7.3.2 Column Buckling with One Free and One Fixed End

This case deals with a column with a free end, where the load is applied, and a fixed end as depicted in Fig. 7.7. A FBD of the free end of the cantilever column is shown in Fig. 7.7d. From this FBD we write the moment equation as:

$$M = P(\delta - y) \qquad (7.25)$$

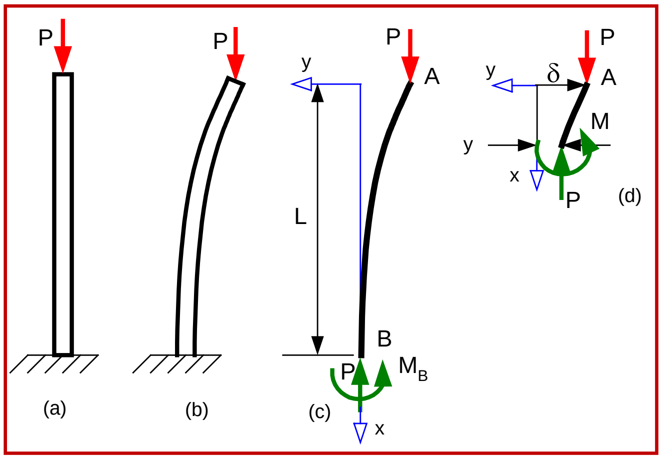


Fig. 7.7 A cantilever type column with one end free and one end fixed.

Substituting Eq. (7.25) into Eq. (7.1) and rearranging to write the differential equation in standard format gives:

$$\frac{d^2y}{dx^2} + p^2 y = p^2 \delta \qquad (7.26)$$

Using similar procedures to solve Eq. (7.26), we obtain:

$$y = y_h + y_p = C_1 \sin(px) + C_2 \cos(px) + \delta \qquad (7.27)$$

Solve for the constants of integration using boundary conditions. Because the top of the column undergoes a deflection δ, we write:

$$y(0) = \delta \qquad (7.28)$$

The boundary conditions for the fixed end are:

$$y(L) = 0 \qquad \frac{dy}{dx}(L) = 0 \qquad (7.29)$$

It is clear from the boundary condition listed in Eq. (7.28) that $C_2 = 0$. Next differentiate Eq. (7.27) with respect to x to obtain:

$$dy/dx = pC_1 \cos(px) \qquad (7.30)$$

Substituting the boundary condition (dy/dx = 0 at x = L) from Eq. (7.29) into Eq. (7.30) gives:

$$pC_1 \cos(pL) = 0 \qquad (7.31)$$

It is evident that Eq. (7.31) is satisfied when $pL = n\pi/2$. The first bucking mode occurs when n = 1; hence, $pL = \pi/2$. Substituting this value into Eq. (7.3) gives the critical buckling load as:

$$P_{CR} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2 EI}{(2L)^2} \qquad (7.32)$$

For this column with free and fixed ends, k = 2. Comparison of Eq. (7.32) with Eq. (7.23) shows a remarkable reduction in the critical buckling load when the pin at the top of the column is removed and an axial compression load is applied to the column's free end. The reduction is a factor of about eight.

7.3.3 Column Buckling with Two Fixed Ends

The final case considered is the column with both ends fixed as shown in Fig. 7.8. The procedure for deriving the Euler equation for the critical load for buckling is similar to that described previously. Begin with the FBD, shown in Fig. 7.8d and write the moment equation as:

$$M = -Py + M_A \qquad (7.33)$$

Substituting Eq. (7.33) into Eq. (7.1) and rearranging gives the differential equation in standard format as:

$$\frac{d^2y}{dx^2} + p^2 y = \frac{M_A}{EI} \qquad (7.34)$$

Solving Eq. (7.34) for the homogeneous and particular solutions gives the general solution as:

$$y = y_h + y_p = C_1 \sin(px) + C_2 \cos(px) + M_A/P \qquad (7.35)$$

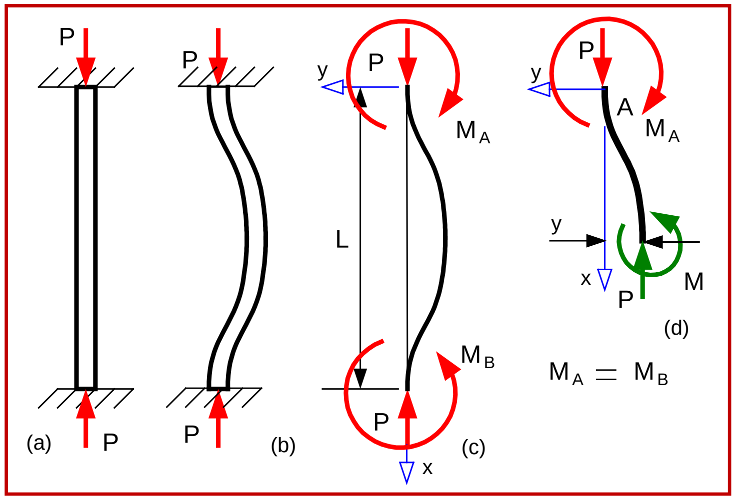


Fig. 7.8 A column with both ends fixed subjected to axial compression loading.

We follow the usual procedure to solve for the constants of integration using boundary conditions. At the top of the column, where x = 0, y = 0 and dy/dx = 0. At the center of the column symmetry exists so that dy/dx = 0 at x = L/2. At the bottom of the column, where x = L, y = 0 and dy/dx = 0. We have five boundary conditions and only two unknown constants of integration C_1 and C_2. Let's use the boundary conditions involving the slope dy/dx at x = 0 and x = L/2.

Differentiating Eq. (7.35) yields:

$$dy/dx = pC_1 \cos(px) - pC_2 \sin(px) \tag{7.36}$$

Substituting the boundary condition (dy/dx = 0 at x = 0) into Eq. (7.36) gives C_1 as:

$$C_1 = 0 \tag{7.37}$$

From the boundary condition (dy/dx = 0 at x = L/2) and Eqs. (7.36) and (7.37), we can write:

$$pC_2 \sin(pL/2) = 0 \tag{7.38}$$

Equation (7.38) is satisfied with either $C_2 = 0$ or with sin (pL/2) = 0. If $C_2 = 0$, y = 0 and the solution precludes buckling and does not yield a valid result. However, the condition leading to buckling is determined from the remaining option:

$$\sin(pL/2) = 0 \tag{7.39}$$

Equation (7.39) is satisfied when (pL/2) = nπ. The first mode of buckling occurs with n = 1, which gives:

$$pL/2 = \pi \tag{7.40}$$

Then substituting Eq. (7.3) into Eq. (7.40), we obtain:

$$P_{CR} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2} \tag{7.41}$$

For this column with two fixed ends, k = 0.5. Let's apply these new relations to determine the critical load for two different columns with various end conditions.

EXAMPLE 7.3

A column 19 feet long is fabricated from a steel tube with the rectangular cross section presented in Fig. E7.3. The column is pinned at one end and fixed at the other. Determine the critical buckling force.

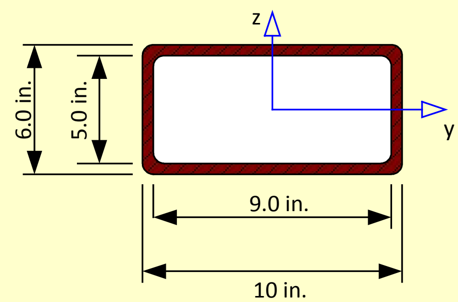


Fig. E7.3

Solution:

We employ Eq. (7.24) to write the expression for the critical buckling force for this column as:

$$P_{CR} = \frac{\pi^2 \ EI}{(kL)^2} \qquad \text{(a)}$$

where k = 0.7 for a column with one end fixed and the other end pinned. The modulus of elasticity $E = 30 \times 10^6$ psi = 30,000 ksi for steel is obtained from Appendix B-1.

The cross section is rectangular; hence the moment of inertia is a function of the choice of axis. For this reason, we will determine both I_y and I_z from Fig. 6.5 as:

$$I_y = (1/12)[b_o h_o^3 - b_i h_i^3] = (1/12)[(10)(6)^3 - (9)(5)^3] = 86.25 \text{ in.}^4 \qquad \text{(b)}$$

$$I_z = (1/12)[h_o b_o^3 - h_i b_i^3] = (1/12)[(6)(10)^3 - (5)(9)^3] = 196.25 \text{ in.}^4 \qquad \text{(c)}$$

Substituting the lower value of the moment of inertia from Eq. (b) into Eq. (a) yields:

$$\left(P_{CR}\right)_y = \frac{\pi^2 \ EI_y}{(0.7)^2 L^2} = \frac{\pi^2 (30{,}000)(86.25)}{(0.7)^2 [(19)(12)]^2} = 1{,}003 \text{ kip} \qquad \text{(d)}$$

Examination of the results indicates the importance of the choice of axis when determining the moment of inertia when the cross section is not circular or square. In this case, the difference in the inertia is about a factor of 2.28.

EXAMPLE 7.4

A column 6.84 m long is fabricated from a steel wide flange section with a designation of W152 × 24. Determine the critical buckling force for the following two cases: (a) the column is free at one end and fixed at the other, and (b) the column is fixed at both ends.

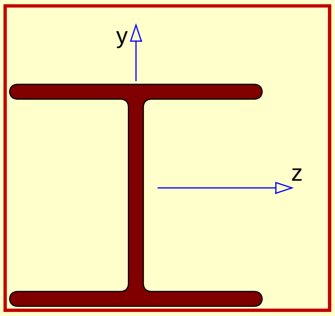


Fig. E7.4

Solution:

Employ Eq. (7.32) to write the expression for the critical buckling force for the column with the top end free as:

$$P_{CR} = \frac{\pi^2 EI}{4L^2} \qquad \text{(a)}$$

The modulus of elasticity (207 GPa) for steel is obtained from Appendix B-1. The moment of inertia is a function of the choice of axis. Both I_y and I_z are given in Appendix C as:

$$I_y = 1.84 \times 10^6 \text{ mm}^4 \qquad I_z = 13.4 \times 10^6 \text{ mm}^4 \qquad \text{(b)}$$

for a wide flange W152 × 24 section. We select the minimum value of the moment of inertia I_y given in Eq. (b) and substitute this value into Eq. (a) to obtain:

$$\left(P_{CR}\right)_y = \frac{\pi^2 EI_y}{4L^2} = \frac{\pi^2 (207)(1.84 \times 10^6)}{(4)(6,840)^2} = 20.09 \text{ kN} \qquad \text{(c)}$$

Next, employ Eq. (7.41) to obtain the expression for the critical buckling force for the column with both ends fixed:

$$\left(P_{CR}\right)_y = \frac{4\pi^2 EI_y}{L^2} = \frac{4\pi^2 (207)(1.84 \times 10^6)}{(6,840)^2} = 321.4 \text{ kN} \qquad \text{(d)}$$

Again, we have selected the minimum value of the moment of inertia I_y for evaluating the critical buckling force. Examination of the results indicates the importance of the choice of the end conditions for column buckling. In this case, the difference in the critical buckling load is a factor of sixteen.

7.3.4 Summary of Equations For Critical Column Buckling Loads

We have derived four equations for the buckling of columns under the action of centric compressive loads. These equations all are of the form $P_{CR} = \pi^2 EI/(kL)^2$. We summarize the results for the parameter k as a function of the various end constraints imposed on the column in Table 7.1

Table 7.1 The constant k as a function of column end constraints

Case No.	**End Constraints**	**k**
1	Pinned—Pinned	1
2	Pinned—Fixed	0.7
3	Free—Fixed	2
4	Fixed—Fixed	0.5

The determination of the critical buckling load for columns subjected to centric compressive forces involves identifying the end constraints and substituting the correct value of k into Eq. (7.24). Care must also be exercised in selecting the correct axis about which to determine the moment of inertia I, because two choices exist in most cases. The minimum inertia usually is employed in the determination. Four examples have been described to demonstrate the method for computing the Euler buckling loads.

The critical buckling stress σ_{CR} has not been emphasized in this section, as we were more concerned with determining the critical buckling force. The topic of stresses produced in columns is discussed in much more detail in the next section.

7.4 COLUMN STRESSES AND LIMITATIONS OF EULER'S THEORY

When a centric load is applied to a column, an axial compressive stress develops that increases until the column becomes unstable. At the inception of buckling, the applied stress is called the critical stress and is given by:

$$\sigma_{CR} = P_{CR}/A \tag{7.42}$$

If we consider the column pinned at both ends, then $P_{CR} = \pi^2 EI/L^2$. Substituting this result into Eq. (7.42) and rearranging symbols yields:

$$\sigma_{CR} = \frac{\pi^2 EI}{L^2 A} = \frac{\pi^2 E(Ar^2)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2} \tag{7.43}$$

Note that $I = Ar^2$ in the expansion of Eq. (7.43), where A is the cross sectional area of the column and r is the radius of gyration of the column's area.

The term (L/r) is known as the slenderness ratio of the column. As (L/r) increases the column becomes more flexible and will buckle at lower stress levels. Let's consider an example to illustrate the reduction in the critical stress σ_{CR} with increasing (L/r).

EXAMPLE 7.5

Consider a steel column with both ends pinned. Note that E = 207 GPa and S_y = 300 MPa. Prepare a graph showing σ_{CR} as a function of the slenderness ratio (L/r).

Solution:

The solution involves evaluating Eq. (7.43) as (L/r) is varied from 10 to 200 in steps of 10, and then comparing the critical stress σ_{CR} to the yield strength S_y of the column material.

$$\sigma_{CR} = \frac{\pi^2 E}{(L/r)^2} \overset{?}{\Leftrightarrow} S_y \qquad \text{(a)}$$

The results obtained from Eq. (a) in a spreadsheet calculation are presented in Fig. E7.5. Examination of this figure indicates that the critical stress σ_{CR} exceeds the yield strength S_y of the steel specified in the example statement for $(L/r) < 82.5$. This fact means that the Euler theory for buckling is valid only when $(L/r) \geq 82.5$. For slenderness ratios less than this value, failure of the column will occur by compressive yielding.

The limiting value of (L/r) depends on the column constraint and both the modulus of elasticity and the yield strength of the material from which the column is fabricated. We will explore this dependency in the next example.

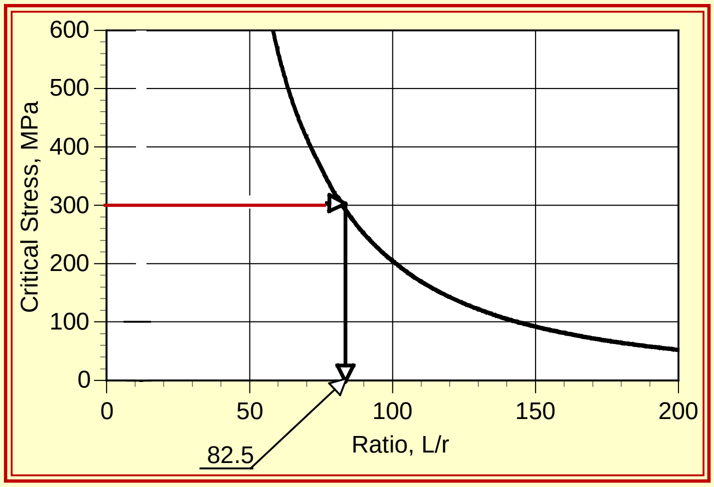


Fig. E7.5 Critical stress σ_{CR} as a function of slenderness ratio (L/r).

EXAMPLE 7.6

Consider a column with both ends pinned. Determine the limiting value of (L/r) as a function of yield strength for columns fabricated from both steel and aluminum. Consider the yield strength of these two materials varying from 20 to 200 ksi.

Solution:

Set $\sigma_{CR} = S_y$ in Eq. (7.43) and solve for (L/r) to obtain:

$$\frac{L}{r} = \sqrt{\frac{\pi^2 E}{S_y}} \qquad \text{(a)}$$

Let $E = 30 \times 10^6$ psi and 10.4×10^6 psi for steel and aluminum, respectively. Use a spreadsheet to evaluate Eq. (a) and vary S_y from 20 to 200 ksi in steps of 10, to obtain the results presented in Fig. E7.6.

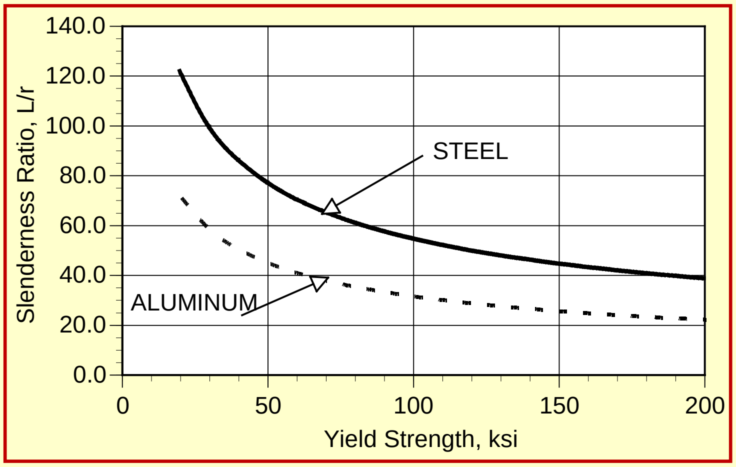


Fig. E7.6 The limiting value of the slenderness ratio as a function of yield strength for a column with both ends pinned.

The marked difference in the limiting value of the slenderness ratio for steel and aluminum alloy is due to the fact that aluminum is less resistant to buckling, because of its much lower modulus of elasticity.

These two examples have illustrated that the Euler theory of buckling has limits of applicability. Long slender columns with high values of (L/r) will buckle under the action of centric compressive loads and the Euler equations are valid. However, if (L/r) for the column is less than the limit value, as is the case for shorter columns with larger radii of gyration, the Euler theory is not valid. In these cases, the column does not buckle. The failure mode is yielding due to excessive compressive stress. The two different modes of failure are illustrated in Fig. 7.9.

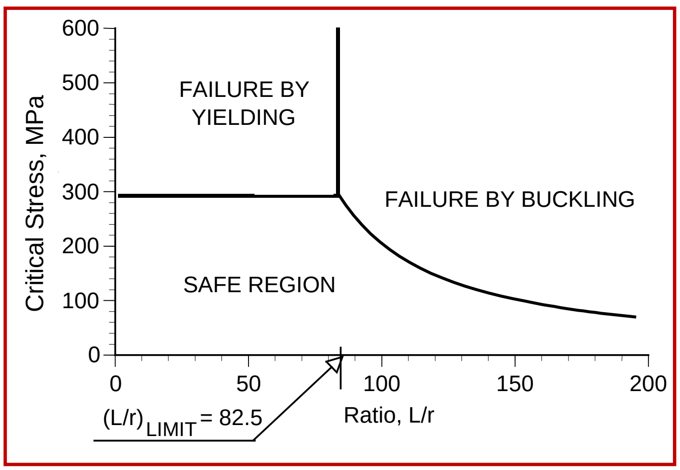
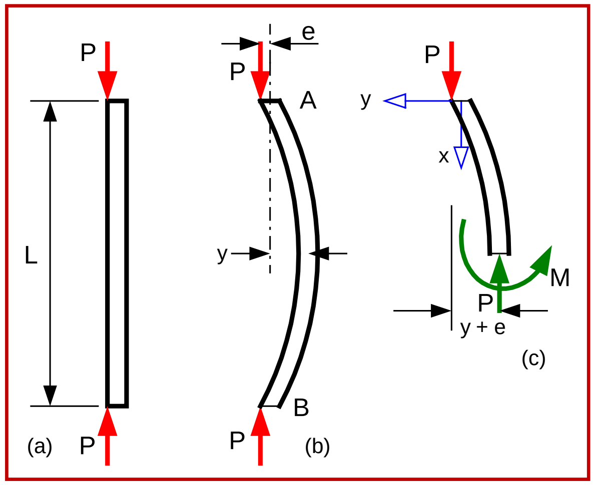


Fig. 7.9 Safety and failure regions for a centric loaded, pin-pin ended column.
Yield strength 300 MPa.

7.5 ECCENTRICALLY LOADED COLUMNS

The developments in the previous sections of this chapter assumed that the axial compressive load was applied to the column through the centroid of its cross section. In practice this centric loading is nearly impossible to achieve. Even if the load is placed at the centroid of the cross section, columns are never perfectly straight. As a consequence the loading always exhibits some small eccentricity e. Let's explore the effect of eccentric loading on the buckling relations, by considering the pin-pin ended column shown in Fig. 7.10. Note that the load P is applied at the edge of the column with an eccentricity relative to the centroid given by the dimension e.

Fig. 7.10 A column with an eccentric axial load P.



The analysis follows the same procedure as described previously. The deformed shape of the column is represented as a sine wave, because the ends are pinned. The column is cut at a location x measured from the top end and a FBD of this end is drawn, as shown in Fig. 7.10c. Writing the equation for the moment M gives:

$$M = -P(y + e) \tag{7.44}$$

Substituting Eq. (7.44) into Eq. (7.1) yields:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{1}{EI}\left(Py + Pe\right) \tag{7.45}$$

Rearranging Eq. (7.45) to write the differential equation in standard format gives:

$$\frac{d^2y}{dx^2} + p^2 y = -p^2 e \tag{7.46}$$

where p is given in Eq. (7.3)

Solving Eq. (7.46) yields:

$$y = y_h + y_p = C_1 \sin(px) + C_2 \cos(px) - e \tag{7.47}$$

We follow the usual procedure to solve for the constants of integration using boundary conditions. Because both ends of the column are pinned, we write:

$$y(0) = 0 \qquad\qquad y(L) = 0 \tag{7.48}$$

From the first boundary condition ($y = 0$ at $x = 0$), it is clear that $C_2 = e$. From the second boundary condition ($y = 0$ at $x = L$), we can show that:

$$C_1 = e[1 - \cos(pL)]/\sin(pL) \tag{7.49}$$

Using the following trigometric identities:

$$1 - \cos(pL) = 2\sin^2(pL/2) \qquad\qquad \sin(pL) = 2\sin(pL/2)\cos(pL/2)$$

the expression for C_1 can be rewritten as:

$$C_1 = e \tan(pL/2) \tag{7.50}$$

Substituting the values for C_1 and C_2 into Eq. (7.47) yields:

$$y = = e[\tan(pL/2)\sin(px) + \cos(px) - 1] \tag{7.51}$$

This relation shows that the amplitude of the deflection of the column prior to the initiation of buckling depends upon the eccentricity e. We will show a graph illustrating this fact later in this section. The maximum value of y occurs at the mid-point of the column where x = L/2. After further algebraic and trigonometric manipulations, we can write the expression for y_{Max} at that location as:

$$y_{Max} = e[\sec(pL/2) - 1] \tag{7.52}$$

The column will buckle when y_{Max} becomes large. For small eccentricity, y_{Max} becomes large when:

$$\sec(pL/2) \to \infty \qquad \Rightarrow \qquad pL/2 = \pi/2$$

From this result and Eq. (7.3), it is clear that:

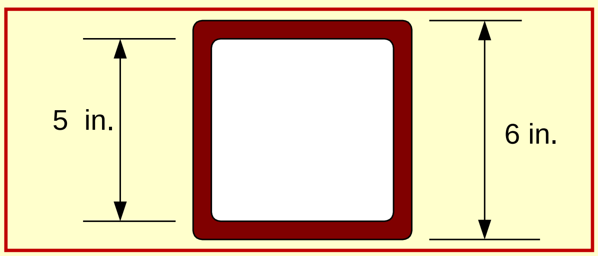
$$P_{CR} = \frac{\pi^2 EI}{L^2} \tag{7.53}$$

The buckling load is identical with that derived for centrically loaded columns with e = 0. While the end result is the same, the bucking process differs. With centric loading, the column remained straight as the load P was increased until P_{CR} was achieved. At the critical load P_{CR}, the column became unstable and suddenly buckled (collapsed). With eccentricity the column deflects laterally with increasing magnitude until the critical load is achieved. We will illustrate this behavior in the next example.

EXAMPLE 7.7

Consider a column with both ends pin loaded with an axial force P. The force P is applied with an eccentricity e. Determine y_{Max} as a function of the axial force P as the quantity pL/2 varies from 0 to π/2. Consider both e = 1 and 2 in. The column is 30 ft long and fabricated from a square steel tube depicted in Fig. E7.7.

Fig. E7.7



Solution:

Before determining y_{Max}, establish the critical buckling load from Eq. (7.53) as:

$$P_{CR} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(30\times10^6)\{[(6)^4-(5)^4]/12\}}{[(30)(12)]^2} = 127,700 \text{ lb} \tag{a}$$

This result enables us to bound the calculations needed in preparing a graph of y_{Max} as a function of P. Recall Eq. (7.52) and rewrite it as:

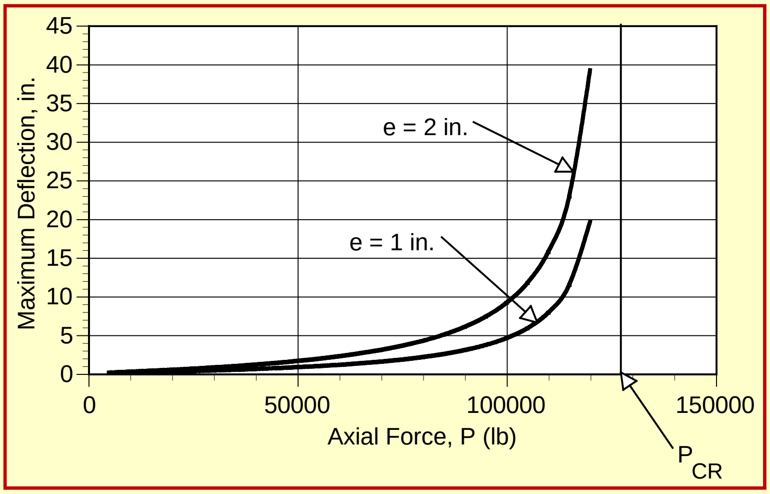
$$y_{Max} = e\left(\sec\left(\frac{pL}{2}\right) - 1\right) = e\left(\sec\sqrt{\frac{PL^2}{4EI}} - 1\right) \qquad \text{(b)}$$

Substituting numerical parameters into Eq. (b) gives:

$$y_{Max} = e\left(\sec\sqrt{\frac{PL^2}{4EI}} - 1\right) = e\left(\sec\sqrt{\frac{P(360)^2}{4(30\times10^6)(55.92)}} - 1\right) = e\left(\sec\sqrt{(19.31\times10^{-6})P} - 1\right) \qquad \text{(c)}$$

Next, employ a spreadsheet to evaluate Eq. (c) for P varying from 0 to 130 kip to obtain the graph shown in Fig. E7.7a.

Fig. E7.7a



Examination of the results in Fig. E7.7a shows that the deflection is a non-linear function of the axial load P. The deflection increases exponentially, as the critical buckling load is approached. This response is due to the non-linear behavior of moment M = P(e + y), because the deflection y increases together with the axial load P.

7.6 STRESSES IN COLUMNS WITH ECCENTRIC LOADING

In some structural applications, it is necessary to apply loads to columns with significant eccentricity. While this practice does not affect the critical buckling load as determined by the Euler theory, it may result in failure of the column due to excessive compressive stress at loads less than P_{CR}. Let's consider the free-ended column, shown in Fig. 7.11, with an axial force P that is applied with an eccentricity e.

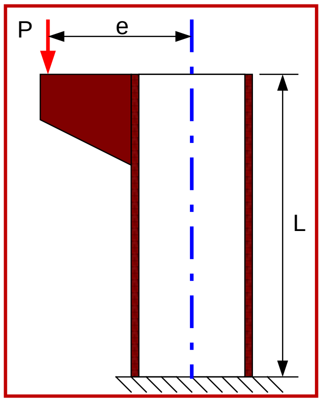


Fig. 7.11 A free-ended column with an eccentric load P.

The load P generates an internal force and a bending moment both of which are constant over the length of the column. A FBD of the top portion of the column, presented in Fig. 7.12, shows the internal force P and moment M.

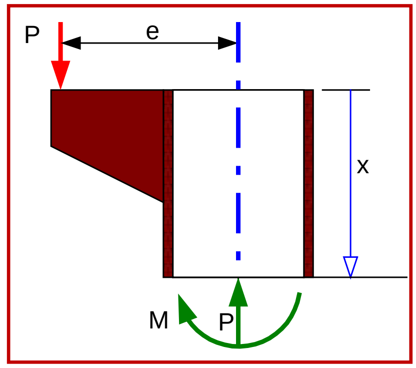


Fig. 7.12 A FBD showing an internal force P and a bending moment M.

Recall from Eq. (7.44) that the bending moment M depends on both the eccentricity e and the lateral deflection y. Considering equilibrium enables us to write:

$$M = P\,(y + e) \qquad (7.54)$$

The axial stresses produced in the column due to P and M[3] are:

$$\sigma_c = \sigma_a + \sigma_b = -\,P/A \pm Mc/I \qquad (7.55)$$

It is evident from Eq. (7.55) that the axial stresses due to the axial force and the bending moment combine. The combined distribution of axial stresses across the column is illustrated in Fig. 7.13.

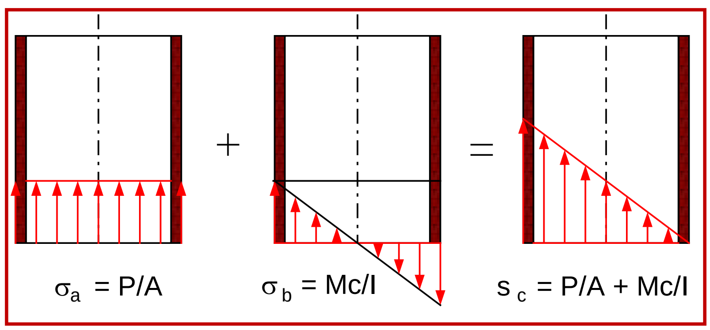


Fig. 7.13 Combined stresses due to eccentric loading of a column.

For an axial compressive force applied to a column, the maximum stress is due to the combined axial and bending stresses given by:

$$\sigma_{Max} = \sigma_a + \sigma_b = -\,(P/A + M_{Max}c/I) \qquad (7.56)$$

Substituting Eq. (7.54) into Eq. (7.56) gives:

$$\sigma_{Max} = -\,[P/A + P(y_{Max} + e)c/I] \qquad (7.57)$$

Using the value for y_{Max} from Eq. (7.52), we can write:

$$\sigma_{Max} = -P\left(\frac{1}{A} + \frac{ce}{I}\sec\left(\frac{pL}{2}\right)\right) \qquad (7.58)$$

[3] Stresses due to the bending moment M will be discussed in the Mechanics II course.

An alternate formula for σ_{Max} can be obtained by substituting Eqs. (7.3) and (7.53) into Eq. (7.58). After simplification:

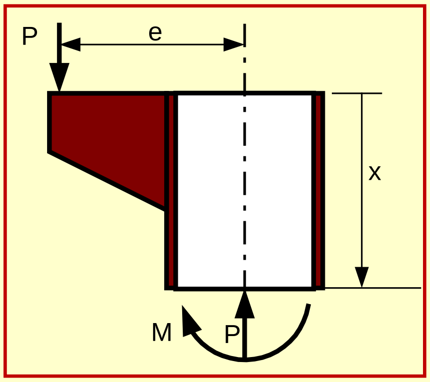
$$\sigma_{Max} = -P\left(\frac{1}{A} + \frac{ce}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{CR}}}\right)\right) \qquad (7.59)$$

Equation (7.59) is the well-known secant formula for eccentrically loaded columns. The equation may be used for columns with any end conditions, as long as the appropriate formula for P_{CR} is selected. Also note that Eq. (7.59) is non-linear in terms of the load P. Thus, it is often necessary to solve the equation by trial and error or by using a programmable calculator or computer software, etc. For the same reason, any safety factor should only be applied to the loads and not to the stresses.

In the design of an eccentrically loaded column, the procedure is to determine the maximum stress using Eq. (7.59) and the critical buckling stress using the appropriate relation that depends on the end conditions of the column. We then compare these two values to determine the mode of failure for the column. This procedure is illustrated in the next example.

EXAMPLE 7.8

Consider a column with both ends pin loaded with an axial force P as shown in Fig. E7.8. The force P is applied with an eccentricity e = 325 mm. Determine the maximum load P that can be applied if a safety factor SF = 3.0 is to be maintained against failure by excessive compressive stress or by buckling. The steel column is 7.4 m long and is fabricated from an American Standard section with a designation of S127 × 22. The steel is 1020 HR with S_y = 315 MPa.



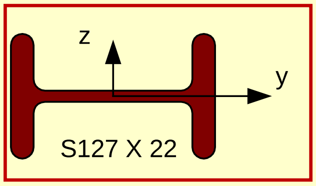


Fig. E7.8

Solution:

We employ Eq. (7.8) to write the expression for the critical buckling force for this pinned ended column as:

$$P_{CR} = \frac{\pi^2 EI}{L^2} \qquad (a)$$

The modulus of elasticity E = 207 GPa for the steel is obtained from Appendix B-1.

The moment of inertia is a function of the orientation of the pin. If the pin is inserted along the y axis (see Fig. E7.8), then the end of the column will rotate about this axis. However, if the pin is inserted in the z direction, the column will rotate about this axis. Both I_y and I_z are given in Appendix C as:

$$I_y = 0.695 \times 10^6 \text{ mm}^4 \qquad I_z = 6.33 \times 10^6 \text{ mm}^4 \tag{b}$$

Substituting the minimum value I_y into Eq. (a) yields:

$$\left(P_{CR}\right)_y = \frac{\pi^2 EI_y}{L^2} = \frac{\pi^2(207)(0.695\times10^6)}{(7,400)^2} = 25.93 \text{ kN} \tag{c}$$

Examination of the results indicates the importance of the direction of the pin when the cross section is not circular or square. In this case, the difference in the moment of inertia I is in excess of a factor of nine. Clearly, for the section shown in Fig. E7.8, the pin should be inserted in the z direction. Many conservative designers will base their recommendations on the minimum critical load, even with the column end free to rotate about the z-axis. Let's follow this conservative approach. Then the allowable load for buckling is given by:

$$(P_{Allowable})_{CR} = P_{CR}/SF = (25.93)/(3.0) = 8.643 \text{ kN} \tag{d}$$

The maximum stress is given by Eq. (7.59) as:

$$\sigma_{Max} = -S_y = -P_y\left(\frac{1}{A} + \frac{ce}{I_y}\sec\left(\frac{\pi}{2}\sqrt{\frac{P_y}{P_{CR}}}\right)\right) \tag{e}$$

Additional properties for the American Standard section are obtained from Appendix C as:

$$A = 2,800 \text{ mm}^2 \tag{f}$$

$$c = w_f/2 = (83.4)/2 = 41.7 \text{ mm} \tag{g}$$

Substituting all numerical parameters for the S127 × 22 section and the yield strength of the 1020 HR steel into Eq. (e) gives:

$$315 = P_y\left(\frac{1}{2,800} + \frac{(41.7)(325)}{0.695\times10^6}\sec\left(\frac{\pi}{2}\sqrt{\frac{P_y}{25,930}}\right)\right) \tag{h}$$

Solving Eq. (h) by trial and error:

$$P_y = 9,371 \text{ N} = 9.371 \text{ kN} \tag{i}$$

Then the allowable load for yielding is:

$$(P_{Allowable})_y = P_y/SF = (9.371)/(3.0) = 3.124 \text{ kN} \tag{j}$$

Compare both allowable loads from Eqs. (d) and (j) and choose the smallest one, which gives:

$$P_{Allowable} = 3.124 \text{ kN} \tag{k}$$

In this case, the column will fail by yielding rather than buckling.

7.7 SUMMARY

The concept of elastic instability was introduced. Columns can fail by buckling at stress levels lower than the yield strength of the materials from which they are fabricated. Failure by buckling is catastrophic as the column collapses suddenly. The critical load P_{CR} depends on the end conditions of the column with the critical load increasing as the constraint at its ends increase.

Equations for P_{CR} were derived for four different end conditions. The results are given by:

$$P_{CR} = \frac{\pi^2 EI}{(kL)^2} \qquad (7.24)$$

where k, given in the table below, accounts for the constraint provided by the column supports.

Case No.	End Constraints	k
1	Pinned—Pinned	1
2	Pinned—Fixed	0.7
3	Free—Fixed	2
4	Fixed—Fixed	0.5

The moment of inertia I of the cross section of the column in Eq. (7.24) is almost always taken as the minimum value.

The critical stress σ_{CR} in centric loaded columns was defined as:

$$\sigma_{CR} = P_{CR}/A \qquad (7.42)$$

The critical stress may be written as:

$$\sigma_{CR} = \frac{\pi^2 EI}{L^2 A} = \frac{\pi^2 E(Ar^2)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2} \qquad (7.43)$$

where A is the cross sectional area of the column and r is the radius of gyration of the area.

The term (L/r) is known as the slenderness ratio of the column. As (L/r) increases, the column becomes more flexible and will buckle at lower stress levels. When the slenderness ratio is large, columns buckle in accordance with Eq. (7.24); however, for shorter and stiffer columns $\sigma_{CR} > S_y$. In these cases the column fails by yielding. The transition from the Euler theory of buckling to yielding occurs at a limit value of (L/r), which depends on the modulus of elasticity and yield strength of the column material as indicated in Example 7.6. Safety and failure regions for a typical column are illustrated in Fig. 7.9.

When the load applied to the column exhibits some eccentricity, a bending moment is produced which affects the stresses in the column. However, the eccentricity does not affect the critical bucking load. While the result for P_{CR} is the same, the bucking process differs. With centric loading the column remains straight as the load P is increased until P_{CR} is reached. At the critical load, the column becomes unstable and suddenly buckles (collapses). With eccentricity, the column deflects laterally with increasing magnitude until the critical load is achieved.

The stress in an eccentrically loaded column is due to a combination of a compressive stress σ_a due to P and a bending stress σ_b due to the moment M = P (y + e). The two stresses superimpose as indicated by:

$$\sigma_{Max} = \sigma_a + \sigma_b = -\left(P/A + M_{Max}c/I\right) \quad (7.56)$$

An alternate formula for σ_{Max} can be derived as:

$$\sigma_{Max} = -P\left(\frac{1}{A} + \frac{ce}{I}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{CR}}}\right)\right) \quad (7.59)$$

This equation, known as the secant formula, may be used for columns with any end conditions by calculating P_{CR} from Eq. (7.24). Because the stress is non-linearly related to the load, it is often necessary to solve the equation by trial and error or by using advanced methods, such as a programmable calculator or computer software. Also, any safety factor should only be applied to the loads and not to the stresses. For eccentrically loaded columns, both the maximum stress and the critical buckling stress must be computed and compared to determine the actual mode of failure.

CHAPTER 8

FRAMES AND MACHINES

8.1 INTRODUCTION

Frames are similar to trusses because they both are fabricated from long thin members. However, there are two important differences.

1. The joints in a truss are pinned and free to rotate; whereas, one or more joints in a frame may be rigid. Because rotation is constrained, the rigid joints are capable of providing reaction moments.
2. The forces applied to a truss act only at the joints; whereas, forces may be applied at any location on a frame. Thus, one or more members in a frame may be subjected to more than two forces.

All of the members in a truss are subjected to only two external forces; hence, the internal force must coincide with the line of action of these two equal and opposite forces. In a frame one or more members may be subjected to more than two external forces. When we make a section cut across a multiforce member[1], it is necessary to assume the existence of an internal axial force P, an internal shear force V, and an internal moment M. Because of this fact (introducing three unknowns for each member cut), the method of joints and the method of sections are usually not effective when analyzing multiforce members.

To analyze a frame, we follow the same general approach as with trusses, but carefully avoid making section cuts on any multiforce member:

1. Prepare a FBD of the entire frame.
2. Apply the appropriate equilibrium relations to solve for the reactions.
3. Dismember the frame and draw FBDs of each member.
4. Use Newton's law of action and reaction in preparing these FBDs.
5. Identify the two-force members (if any), and show the two equal and opposite forces acting at the pinned joints.
6. Apply the appropriate equilibrium relations to solve for the internal forces and moments.

Machines are similar to frames in that they contain one or more multiforce members. However, machines differ from both frames and trusses, because motion occurs as they act to convert an input force (or moment) to an output force (or moment). Machines may act to amplify or attenuate forces or change their direction. Simple machines include the lever, screw, inclined plane and pulley. While machines may amplify or attenuate forces as these forces are transmitted from the machine's input to its output, energy is not gained in the process. Energy is conserved except when heat is generated due to frictional effects. This heat is usually dissipated into the atmosphere.

The approach for solving problems involving frames and machines will be demonstrated in the following examples. The importance of drawing complete and accurate FBDs is stressed in these examples.

[1] A multiforce member is subjected to three or more external forces, or alternatively two or more external forces and one or more external moments.

8.2 FRAMES — EXAMPLES

EXAMPLE 8.1

To demonstrate the method for analyzing frames, consider the hoist presented in Fig. E8.1. First, identify the multiforce and two-force members. Then solve for the reactions at points A and E. Determine all of the external forces acting on member CFG. The force F on the hoist is 1,000 N. Point E is a pinned support and point A is a roller support.

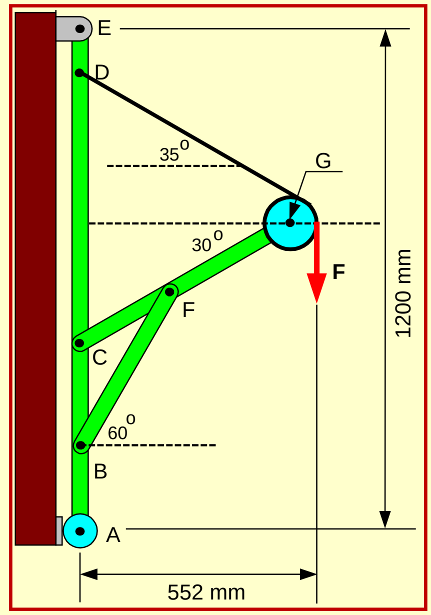


Fig. E8.1

Solution:

Step 1: Identify the multiforce members and two force members.

The vertical bar ABCDE and the inclined bar CFG are multiforce members. The cable DG and the strut BF are two-force members.

Step 2: Draw a FBD of the entire frame as shown in Fig E8.1a.

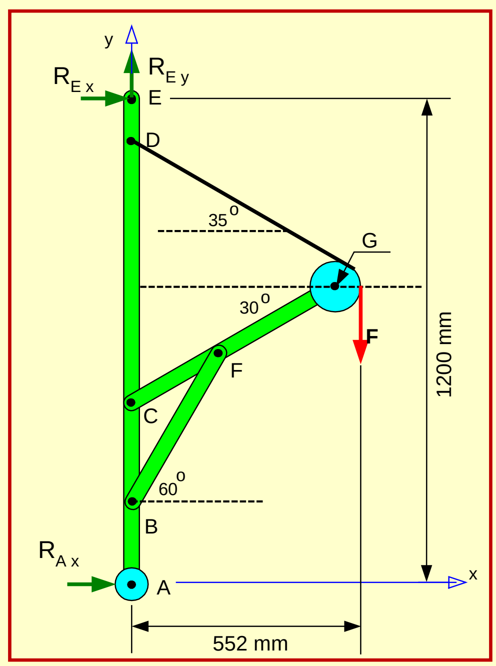


Fig. E8.1a

Step 3: Let's employ the equilibrium relations to determine the reactions at points A and E.

$$\Sigma M_E = 1{,}200\, R_{Ax} - 552\, F = 0 \quad \Rightarrow \quad R_{Ax} = (0.46)(1{,}000) = 460.0 \text{ N} \qquad \text{(a)}$$

$$\Sigma F_x = R_{Ax} + R_{Ex} = 0 \quad \Rightarrow \quad R_{Ex} = -R_{Ax} = -460.0 \text{ N} \qquad \text{(b)}$$

$$\Sigma F_y = R_{Ey} - F = 0 \quad \Rightarrow \quad R_{Ey} = F = 1{,}000 \text{ N} \qquad \text{(c)}$$

Step 4: Next, prepare the FBDs of the individual members as shown in Fig. E8.1b. Four FBDs are presented in this illustration to show the forces acting on members ABCDE, BF, CFG and the pulley at G. Note that we have used the law of action and reaction in preparing these FBDs. Let's examine member CFG to ascertain if it is possible to determine the forces applied to it. We have five forces acting on this member: R_{Cx}, R_{Cy}, R_{FB}, R_{Gx} and R_{Gy}. The force system is coplanar and non-concurrent; hence, only three equilibrium equations are applicable. Because we have five unknowns and only three equations, it is necessary to write two additional equations in terms of two of the unknown forces. We can write these additional equations by considering the equilibrium of the pulley located at point G.

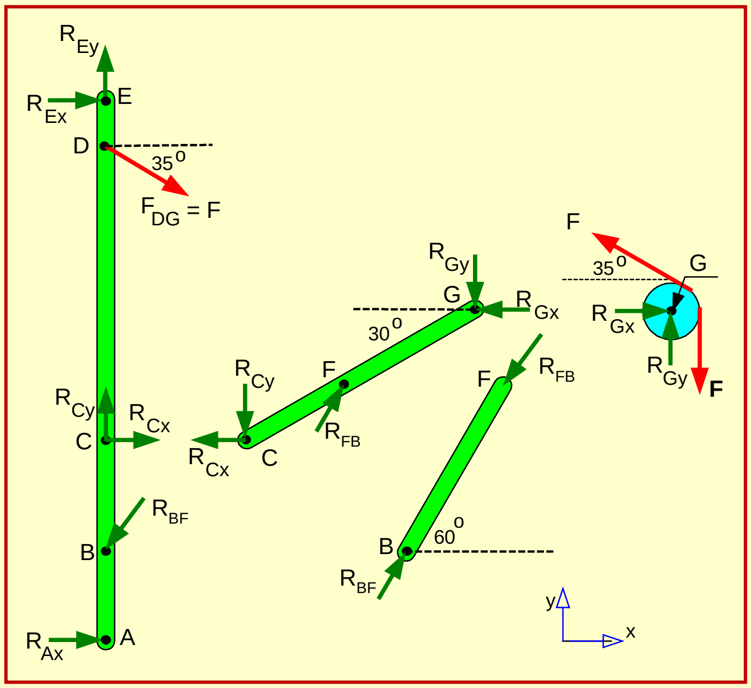


Fig. E8.1b

Step 5: Writing the equilibrium equations for the pulley at point G gives:

$$\Sigma F_x = R_{Gx} - F\cos(35°) = 0 \quad \Rightarrow \quad R_{Gx} = (1{,}000)\cos(35°) = 819.2 \text{ N} \qquad \text{(d)}$$

$$\Sigma F_y = R_{Gy} + F\sin(35°) - F = 0 \quad \Rightarrow \quad R_{Gy} = (1{,}000)[1 - \sin(35°)] = 426.4 \text{ N} \qquad \text{(e)}$$

Step 6: Now it is possible to solve for the forces acting on member CFG from the equilibrium relations. Dimensions were not included in the FBDs presented in Fig E8.1b to avoid confusing the already crowded drawing. Note for member CFG that the distance from point C to point F is 240 mm, from point F to point G is 384 mm and from point C to point G is 624 mm. Therefore, the pulley G radius is 11.6 mm.

$$\Sigma M_C = R_{FB}\,[(240)\sin(30°)] + R_{Gx}\,[(624)\sin(30°)] - R_{Gy}\,[(624)\cos(30°)] = 0 \qquad \text{(f)}$$

Substituting the results from Eqs. (d) and (e) into Eq. (f) yields:

$$R_{FB} = -\,209.7\ \text{N} \qquad \text{(g)}$$

The negative sign indicates that the direction of the force R_{FB} in the FBDs is not correct. We showed R_{FB} as a compressive force in the FBD of members BF and CFG. In reality it is a tensile force.

Continuing with the equilibrium analysis of member CFG and using previous results leads to:

$$\Sigma\, F_x = R_{FB}\cos(60°) - R_{Cx} - R_{Gx} = 0 \quad \Rightarrow \quad R_{Cx} = -\,924.1\ \text{N} \qquad \text{(h)}$$

$$\Sigma\, F_y = R_{FB}\sin(60°) - R_{Cy} - R_{Gy} = 0 \quad \Rightarrow \quad R_{Cy} = -\,608.0\ \text{N} \qquad \text{(i)}$$

We have completed the analysis of member CFG. It is possible to continue the application of the equilibrium relations to the member ABCDE to obtain results for R_{BF}, R_{Cx} and R_{Cy} that will enable you to check the results from the analysis of member CFG.

The Effect of Rigid Joints in Frames

Let's explore the effect of converting the pinned joints to rigid ones on stability of a structure. A rectangular frame with pinned joints is illustrated in Fig. 8.1a. Clearly, the application of a horizontal force at the top left corner of the frame causes it to rotate about its pinned joints. A small horizontal force component will cause the rectangular frame to collapse. However, if the pins at the upper corners are replaced with gusset plates, which are sufficiently rigid to constrain rotation, as shown in Fig 8.1b, the rectangular frame is stable.

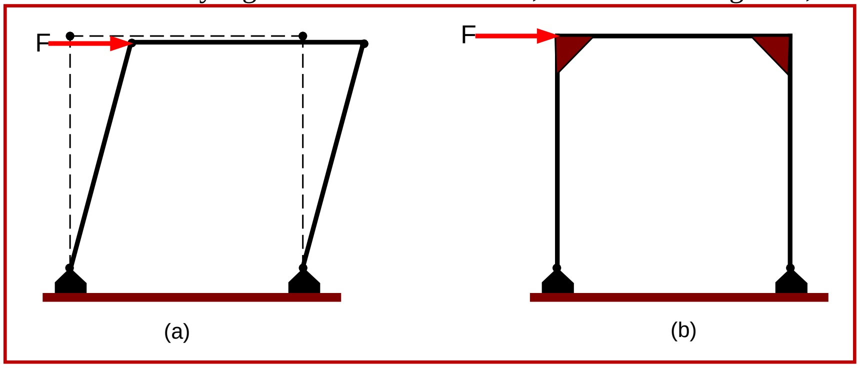


Fig. 8.1 Stability of a rectangular frame is achieved by making the joints rigid.

The forces applied to a frame need not be restricted to the pins location, as the forces can be applied at any point along the length of any member. Of course the application of an additional force along its length produces a multiforce member, as described previously. A rectangular frame with a uniformly distributed force system along the top crossbar is presented in Fig. 8.2.

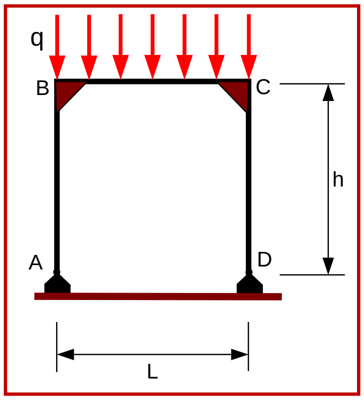


Fig. 8.2 Rectangular frame supporting a uniformly distributed force q.

Let's analyze this rectangular frame to determine the internal loads acting within its members.

EXAMPLE 8.2

Determine the internal forces and moments for the three members in the rectangular frame shown previously in Fig. 8.2.

Solution:

Step 1: The procedure for determining the internal forces and moments is identical to that previously established when analyzing trusses. However, it is critical to recognize that all of the members in the frame will be multiforce members. We begin with a FBD of the entire frame, presented in Fig. E8.2, and then write the appropriate equations of equilibrium.

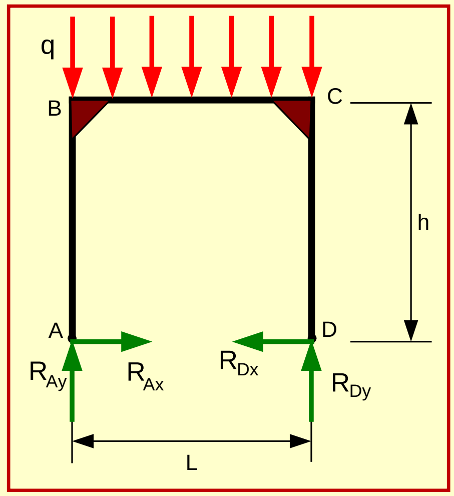


Fig. E8.2

Step 2: The force system in Fig. E8.2 is coplanar and non-concurrent. Accordingly, we write:

$$\Sigma F_x = R_{Ax} - R_{Dx} = 0$$

$$\Sigma F_y = R_{Ay} + R_{Dy} - qL = 0 \qquad \text{(a)}$$

$$\Sigma M_A = R_{Dy} L - (qL)(L/2) = 0$$

Solving Eqs. (a) yields:

$$R_{Ay} = R_{Dy} = qL/2 \qquad \Rightarrow \qquad R_{Ax} = R_{Dx} \qquad \text{(b)}$$

Step 3: Let's draw FBDs for each of the members in the frame, as shown in Fig. E8.2a. In preparing each of these FBDs, we make use of the law of action and reaction. Examination of the FBDs again indicates that the force system is coplanar and non-concurrent for each of its three members.

Step 4: Writing the equations of equilibrium and recognizing the symmetry of both the loading and the geometry yields:

$$M_B = M_C \quad \Rightarrow \quad R_{Bx} = R_{Cx} = R_{Ax} = R_{Dx} \quad \Rightarrow \quad R_{By} = R_{Cy} = R_{Ay} = R_{Dy} = qL/2 \qquad \text{(c)}$$

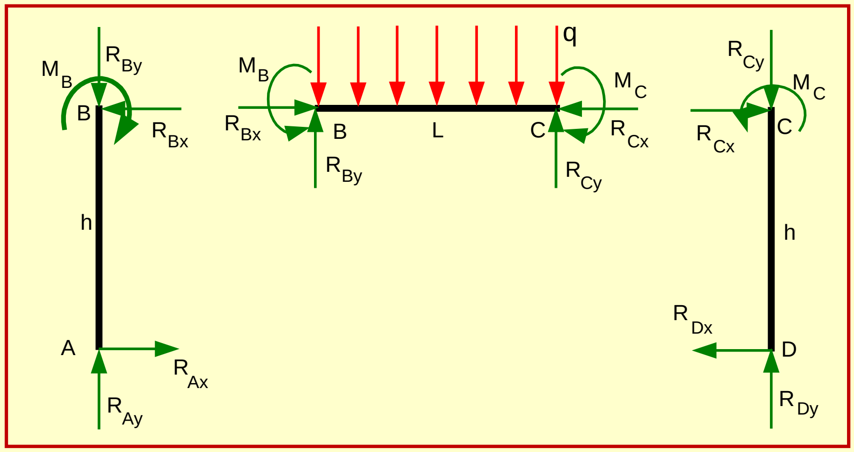


Fig. E8.2a

Consider vertical member AB in Fig. E8.2a and sum moments about point B to give R_{Ax} as:

$$R_{Ax} = M_B/h \qquad \text{(d)}$$

We have employed all of the equations of equilibrium for each of the three members; however, at points B and C, we have not determined the magnitude of the reactions in the x direction or the magnitude of the moment produced by the rigid constraint. We have established the relation given in Eq. (d), but not the magnitude. The problem is statically indeterminate and cannot be solved with only equilibrium relations. To resolve the indeterminacy, it is necessary to consider the deformed shape of the frame, which is beyond the scope of this textbook.

EXAMPLE 8.3

A manual hydraulic pump with a 1.25 in. diameter piston is illustrated in Fig. E8.3. The fluid in the cylinder is pressurized when a force F is applied to the lever arm ADB. If the cylinder pressure is 250 psi, determine the force F applied to the lever arm and the force in the link CD. Note that CD is a two-force member; hence, the internal force F_{CD} is directed along the axis of the link.

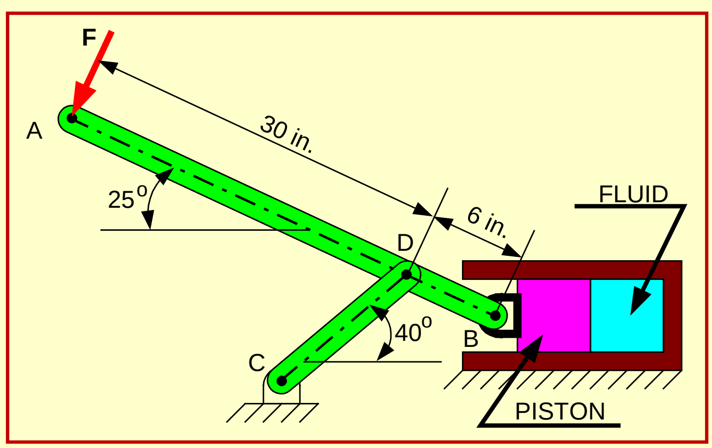


Fig. E8.3

Solution:

Step 1: We begin with a FBD of the entire pump, presented in Fig. E8.3a, and then write the appropriate equations of equilibrium. We neglect the friction forces that act between the piston and the cylinder.

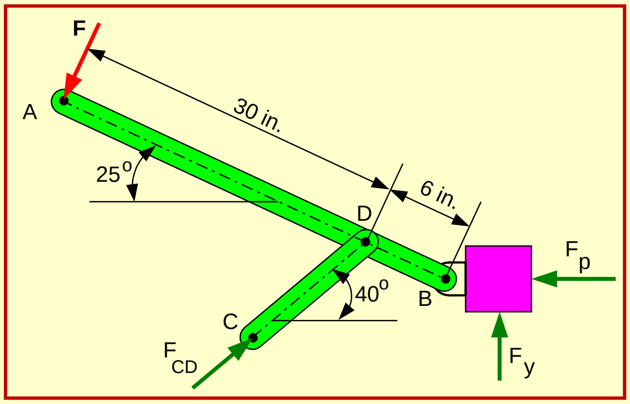


Fig. E8.3a

Step 2: The force F_p acting on the piston is given by:

$$F_p = pA = p\,(\pi d^2/4) = (250)[\pi\,(01.25)^2/(4)] = 306.8 \text{ lb} \qquad \text{(a)}$$

Step 3: The force system in Fig. E8.3a is coplanar and non-concurrent. Accordingly, we write:

$$\Sigma F_x = F_{CD} \cos(40°) - F \sin(25°) - F_p = 0 \qquad \text{(b)}$$

$$\Sigma F_y = F_y + F_{CD} \sin(40°) - F \cos(25°) = 0 \qquad \text{(c)}$$

We cannot write a meaningful moment equation for the FBD in Fig. 8.3a because the location of the force F_y along the piston is not known.

Substituting Eq. (a) into Eq. (b) yields:

$$(0.7660)\,F_{CD} - (0.4226)\,F - 306.8 = 0 \qquad \text{(d)}$$

Step 4: Let's draw FBDs for the two members in the pump's lever system, as shown in Fig. E8.3b. In preparing each of these FBDs, the law of action and reaction is used. Examination of the FBD for the lever arm indicates that the force system is coplanar and non-concurrent.

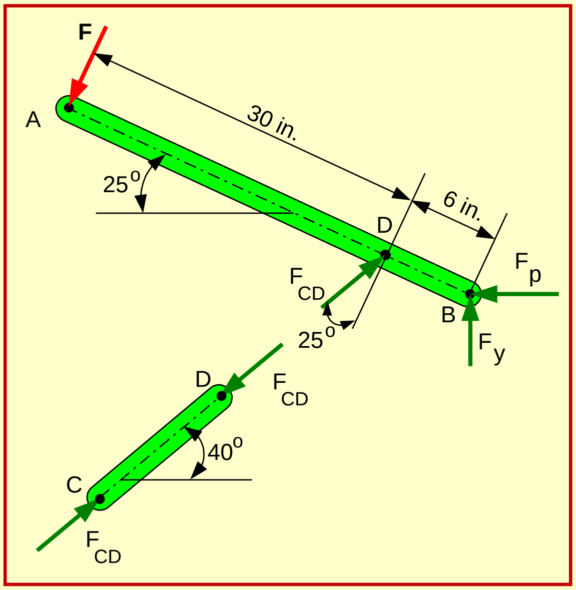


Fig. E8.3b

Step 5: Writing the moment equation of equilibrium for the lever arm about point B yields:

$$\Sigma M_B = (36.00)\,F - (6.00)[F_{CD} \cos(25°)] = 0 \qquad \text{(e)}$$

$$F = (1/6)[F_{CD} \cos(25°)] = 0.1511\, F_{CD} \qquad \text{(f)}$$

Combining Eqs. (d) and (f) gives:

$$F_{CD} = \frac{306.8}{[0.7660 - (0.4226)(0.1511)]} = 436.9 \text{ lb} \qquad \text{(g)}$$

$$F = (0.1511)(436.9) = 66.02 \text{ lb} \qquad \text{(h)}$$

The force required on the lever to develop the pressure of 250 psi is a reasonable value for a manual pump. Clearly, most people would be capable of generating a pump pressure 250 psi with the 36 in. long lever incorporated in this pump's design.

8.3 MACHINES — EXAMPLES

Machines are similar to frames in that they usually contain one or two multiforce members. The difference is that one or more of the members comprising the machine moves. This movement is necessary to amplify or attenuate a force or to change its direction. The equilibrium equations employed in the analysis of machines are identical to those used in solving problems related to trusses and frames. The motion that occurs in these machines is considered to be at constant velocity without acceleration; hence, the static equations of equilibrium apply. As with frames, the analysis usually requires a number of carefully prepared and accurate FBDs.

EXAMPLE 8.4

A pair of pliers clamps a small-diameter, firm-rubber cylinder, as shown in Fig. E8.4. If opposing forces F, are applied to the handles of the pliers increase linearly from zero to 20 lb, determine the force acting on the rubber cylinder. Also determine the work performed if each handle moves though a distance of 0.20 in. Finally, determine the diametrical compression of the rubber cylinder by the application of the forces.

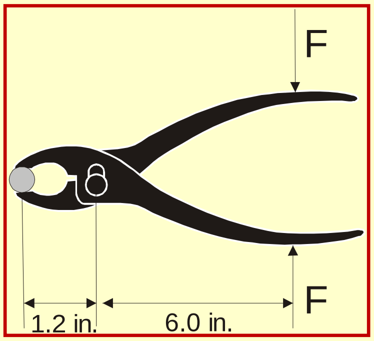
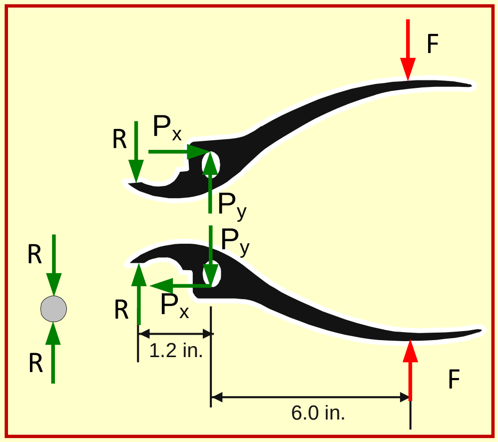


Fig. E8.4

Solution:

Let's consider each member of the pliers individually and prepare a FBD for each as illustrated in Fig. E8.4a. The force acting on the rubber cylinder is R and the force acting at the pivot pin on the pliers is P.

Fig. E8.4a



Three forces are applied to each handle of the pliers; hence the handles are considered as multiforce members. Writing the moment equation about the pivot pin corresponding to the maximum forces, we obtain:

$$\Sigma M_P = (1.2)\,R - (6.0)\,F = 0 \quad \Rightarrow \quad R = (6.0/1.2)\,F = (5)(20) = 100 \text{ lb} \qquad \text{(a)}$$

The pliers exhibit a mechanical advantage of 5, because the output force R is five times the input force F. The mechanical advantage **MA** is given by:

$$\mathbf{MA} = R/F \qquad (8.1)$$

The work $\mathbf{W}$ performed is given by:

$$\mathbf{W} = (F_{Max}\, d)/(2) \qquad (8.2)$$

where d is the distance that the force F is moved along its line of action by a machine.

In this case, each force is moved a distance 0.20 in. along the line of action of F; hence:

$$\mathbf{W} = 2\,(F_{Max}\, d)/(2) = (20)(0.20) = 4.0 \text{ in.-lb} \qquad \text{(b)}$$

Each handle of the pliers is a lever (curved to better fit the hand but a lever nevertheless). The pin holding the two handles together is the fulcrum. The diametrical compression of the rubber cylinder Δd is given by:

$$\Delta d = (1/MA)\, s_h \qquad \text{(c)}$$

$$\Delta d = (1/5)(2)(0.20) = 0.080 \text{ in.} \qquad \text{(d)}$$

where s_h is the distance through which each handle moves.

EXAMPLE 8.5

Determine the reaction force at point G and the mechanical advantage of the toggle mechanism illustrated in Fig. E8.5. The dimensions are a = 310 mm, b = 220 mm, c = 160 mm, d = 100 mm, e = 140 mm and f = 120 mm. The input force F = 150 N. The contact at point G is made with a sphere and pins are inserted at joints B, C, D and E.

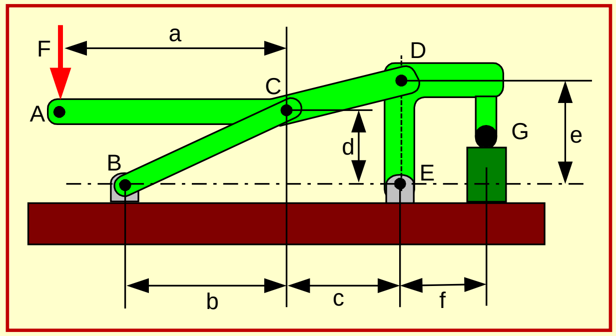


Fig. E8.5

Solution:

Step 1: Identify the link BC as a two-force member and ACD and EDG as multiforce members.

Step 2: Prepare FBDs of the three members, as presented in Fig. E8.5a. Note that we have used the fact that the link BC is a two-force member. The force R_B acts along its axis. Also the reaction at pin C is equal and opposite of the reaction at pin B. We have used the laws of action and reaction in assigning forces and directions at all of the pins. Select member ACD for analysis, because only three unknown forces are acting on it.

Step 3: Determine the angle θ, defined in Fig. E8.5a, and then apply the equations of equilibrium to member ACD.

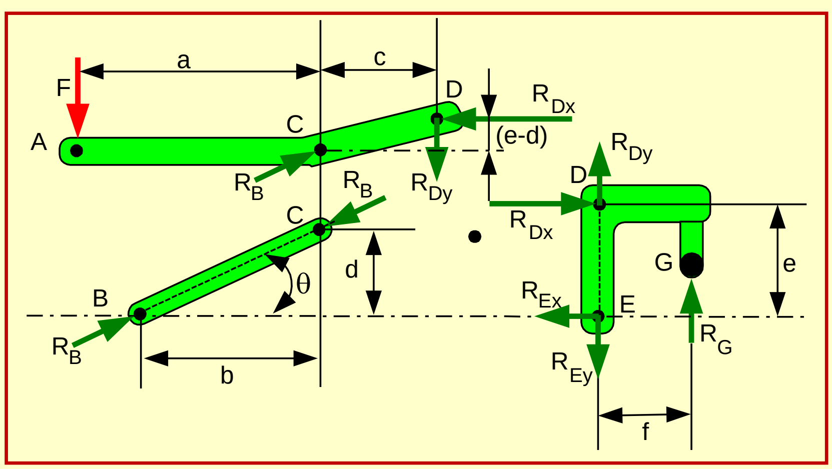
$$\theta = \tan^{-1}(d/b) = \tan^{-1}(100/220) = 24.44° \qquad \text{(a)}$$

$$\Sigma F_x = R_B \cos\theta - R_{Dx} = 0 \quad \Rightarrow \quad R_{Dx} = 0.9104\, R_B \qquad \text{(b)}$$

$$\Sigma F_y = R_B \sin\theta - R_{Dy} - F = 0 \quad \Rightarrow \quad R_{Dy} = 0.4137\, R_B - 150 \qquad \text{(c)}$$

$$\Sigma M_C = F\,(a) - R_{Dy}\,(c) + R_{Dx}\,(e - d) = 0 \qquad \text{(d)}$$

Fig. E8.5a



Substituting the results from Eqs. (b) and (c) into Eq. (d) yields:

$$\Sigma M_C = (150)(310) - (0.4137\ R_B - 150)(160) + (0.9104\ R_B)(40) = 0 \qquad \text{(e)}$$

Solving Eq. (e) for R_B and substituting the result into Eqs. (b) and (c) yields:

$$R_B = 2{,}368\ \text{N} \quad \Rightarrow \quad R_{Dx} = 2{,}156\ \text{N} \quad \Rightarrow \quad R_{Dy} = 829.6\ \text{N} \qquad \text{(f)}$$

Step 4: Apply one of the equations of equilibrium to member EDG.

$$\Sigma M_E = -R_{Dx}\,(e) + R_G\,(f) = 0 \qquad \text{(g)}$$

$$R_G = (e/f)\,R_{Dx} = [(140)/(120)](2{,}156) = 2{,}515\ \text{N} \qquad \text{(h)}$$

Step 5: Determine the mechanical advantage from Eq. (8.1).

$$\mathbf{MA} = R_G/F = (2{,}515)/(150) = 16.77 \qquad \text{(i)}$$

The toggle mechanism has amplified the input force of 150 N to provide an output force of 2,515 N. The mechanical advantage of a toggle mechanism is significant. For this reason, toggles are often used when input forces are limited and high output forces are necessary.

Fig. 8.3 Hydraulic cylinders actuate the arms and bucket on a backhoe.

8.4 CONSTRUCTION EQUIPMENT

Construction equipment provides several real examples of machines that are used to modify forces. The backhoe, shown in Fig. 8.3, involves two large arms that support a bucket. Two hydraulic cylinders are employed to rotate its arms and another hydraulic cylinder is used to rotate the bucket. Both ends of each hydraulic cylinder are pinned, and the ends of each of its large arms arm are pinned. The bucket rotation involves a linkage arrangement.

EXAMPLE 8.6

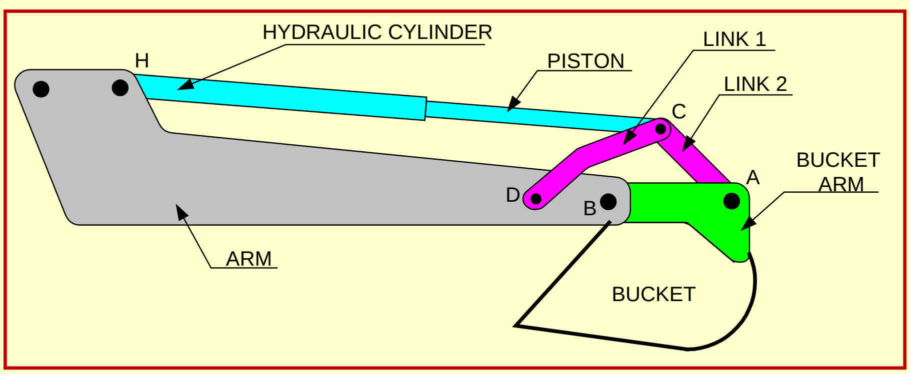
A backhoe, in action, is illustrated in Fig. E8.6. Prepare a drawing showing the bucket, its linkage, the large arm supporting the bucket, and the hydraulic cylinder, which actuates the bucket. Draw FBDs of this machine showing the forces at the pins. If the bucket contains 1,200 N of sand, determine the force exerted by the hydraulic cylinder. The dimensions of the components are presented in the FBDs given in the solution.

Fig. E8.6 A backhoe bucket containing 1,200 N of sand.

Solution:

Step 1: Prepare a drawing showing the arm and linkages supporting the bucket as presented in Fig. E8.6a.

Fig. E8.6a Note the bucket arm and the bucket are welded together.



An examination of this drawing shows that Link 1 and 2 are both two force members and that the hydraulic actuator is also a two-force member. This fact will be instrumental in solving for the force generated by the hydraulic cylinder in supporting the loaded bucket.

Step 2: Prepare a FBD of the backhoe bucket as shown in Fig. E8.6b.

Step 3: Apply the equilibrium relations and solve for the three unknown forces.

$$\Sigma M_B = (405)(F_{AC} \sin(45°)) - (135)(1{,}200) = 0 \quad \text{(a)}$$

$$\Sigma F_y = F_{By} + F_{AC} \sin(45°) - 1{,}200 = 0 \quad \text{(b)}$$

$$\Sigma F_x = F_{Bx} - F_{AC} \cos(45°) = 0 \quad \text{(c)}$$

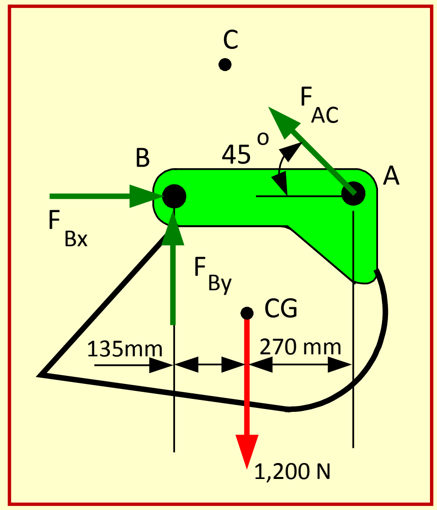


Fig. E8.6b The link between points A and C is a two-force member; hence, the direction of F_{AC} is known.

Solving Eqs. (a), (b) and (c) for the three unknown forces yields:

$$F_{AC} = 565.7 \text{ N} \qquad F_{By} = 800.0 \text{ N} \qquad F_{Bx} = 400.0 \text{ N} \quad \text{(d)}$$

Step 4: Prepare a FBD for the pin at point C, where links 1 and 2 join the piston end of the hydraulic actuator as indicated in Fig. E8.6c.

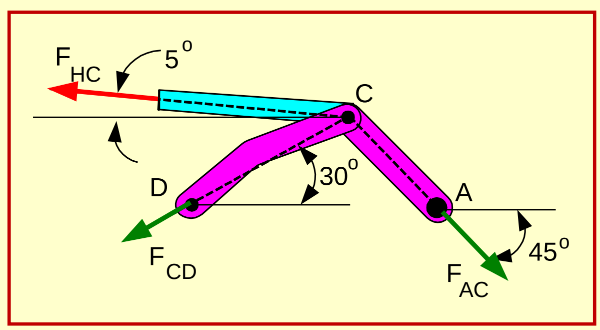


Fig. E8.6c

Step 5: Apply the two equilibrium relations for the concurrent force system and solve for the two unknown forces F_{CD} and F_{HC}.

$$\Sigma F_y = F_{HC} \sin(5°) - F_{AC} \sin(45°) - F_{CD} \sin(30°) = 0 \quad \text{(e)}$$

$$\Sigma F_x = -F_{HC} \cos(5°) + F_{AC} \cos(45°) - F_{CD} \cos(30°) = 0 \quad \text{(f)}$$

Combining the results in Eq. (d) with Eqs. (e) and (f) gives:

$$F_{HC}\,(0.08716) - F_{CD}\,(0.5000) = (565.7)(0.7071) = 400.0 \qquad \text{(g)}$$

$$F_{HC}\,(0.9962) + F_{CD}\,(0.8660) = (565.7)(0.7071) = 400.0 \qquad \text{(h)}$$

Solving these two equations for the two unknowns yields:

$F_{CD} = -\,634.0$ N $\qquad\qquad$ $F_{HC} = 952.6$ N

Of course, the negative sign for F_{CD} indicates the force is compressive.

EXAMPLE 8.7

The Loadall is a versatile piece of construction equipment that can be used for several different purposes. The unit shown in Fig. E8.7 is fitted with a pair of forks for lifting pallets. These forks are fixed to the end of a telescoping boom, and are positioned over a dumpster in the photograph of Fig E8.7. If a pallet weighing 2,000 lb were lifted from a truck bed with these forks, determine the actuator force necessary to rotate the boom and the forces at the bracket hinge pin.

Fig. E8.7 A Loadall unit equipped with a pair of forks for lifting pallets.

Solution:

Step 1: Prepare a drawing showing the telescoping arm and system supporting the forks as presented in Fig. E8.7a.

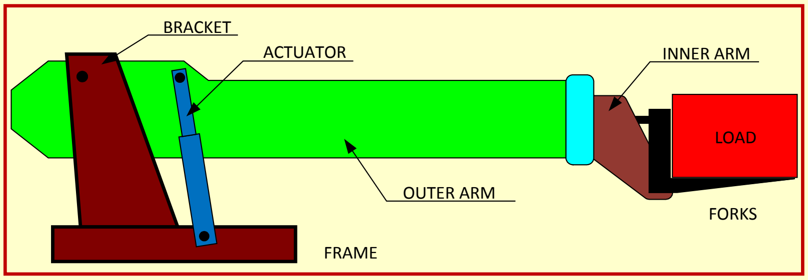


Fig. E8.7a

Step 2: Prepare a FBD of the Loadall mechanism, as shown in Fig. E8.7b. Dimensions of the points where forces are applied are shown in the illustration. The weight of the inner and outer arms acts through the combined center of gravity as shown in Fig. E8.7b.

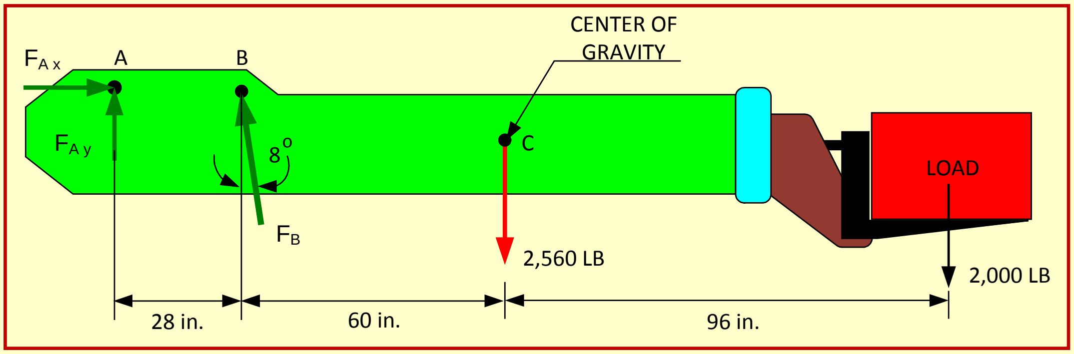


Fig. E8.7b

Step 3: Apply the equilibrium relations and solve for the unknown forces: F_{Ax}, F_{Ay} and F_B.

$$\Sigma M_A = [F_B \cos(8°)](28) - (2{,}560)(88) - (2{,}000)(184) = 0 \qquad \text{(a)}$$

$$F_B = (593{,}280)/[(0.9903)(28)] = 21{,}397 \text{ lb} \qquad \text{(b)}$$

$$\Sigma F_x = F_{Ax} - F_B \sin(8°) = 0 \qquad F_{Ax} = (21{,}397)(0.1392) = 2{,}978 \text{ lb} \qquad \text{(c)}$$

$$\Sigma F_y = F_{Ay} + F_B \cos(8°) - 2{,}560 - 2{,}000 = 0 \qquad \text{(d)}$$

$$F_{Ay} = -(21{,}397)(0.9903) + 4{,}560 = -16{,}629 \text{ lb} \qquad \text{(e)}$$

While the Loadall appears to be a complex machine capable of lifting and moving loads, it is relatively easy to analyze. The arm is a multiforce member, but if the FBD is prepared correctly, the solution for the actuator force F_B and the forces F_{Ax} and F_{Ay} at the hinge pin is straightforward.

8.5 SUMMARY

The method of analysis of frames and machines was introduced. Frames and machines contain one or more members that have three or more externally applied forces. For these multiforce members, the internal force does not coincide with the axis of the member. This fact is extremely important, because it implies that two internal forces (P and V) and an internal moment (M) must be applied when making a section cut through a multiforce member. Because so many unknowns are introduced, section cuts through a multiforce member are usually avoided when analyzing frames and machines. A better approach is to construct FBDs of the individual members of the frame or machine, and to apply the law of action and reaction as well as the equations of equilibrium to each member. Examples for both frames and machines were presented to demonstrate the method of analysis. Examples of construction equipment were described to show that complex machines could be analyzed using these relatively straightforward techniques.

CHAPTER 9

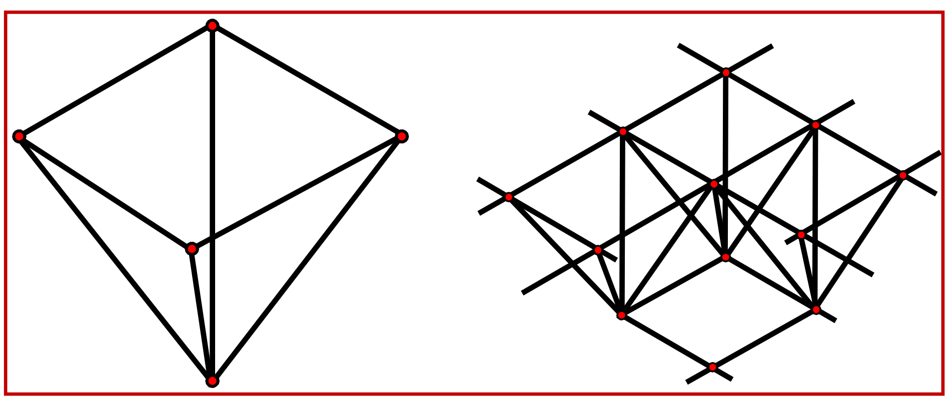
SPACE STRUCTURES AND 3-D EQUILIBRIUM

9.1 INTRODUCTION

A space structure is a three-dimensional system, fabricated from uniaxial members, that spans openings in three-directions. The space structure incorporates members that extend into space with force components in the x-y, x-z and y-z planes. Space structures may be fabricated with rigid joints, which support moments, or pinned joints, which do not. In this chapter, we will first consider space structures with ball and socket joints that cannot support moments. Then we will study more complex three-dimensional structures. Trusses, described in Chapter 5, are two-dimensional structures, because all of the members involved in their construction lie in the x-y plane. Trusses span in only one direction, across a river, highway or a railroad. Design analyses of two-dimensional trusses are performed using only two, or in some cases three, of the six equilibrium relations. However, the three-dimensional nature of space structures is more complex, requiring the repeated application of the six equations of equilibrium to determine the forces in its uniaxial members.

The geometry of space structures is quite diverse with domes, tetrahedrons and half-octahedrons. An example of a space structure constructed with the half octahedrons is presented in Fig. 9.1. Space structures are often employed to support horizontal, flat-roofs that cover large buildings. However, they may be employed in other applications including walls and sloped roofs. Space structures are efficient, because they cost less than a number of trusses spanning the longitudinal direction with beams (purlins) spanning in the transverse direction.

Fig. 9.1 Half-octahedrons arranged to form a space structure.



Another illustration of a space structure is a communications tower that is an extremely tall structure usually built at a high elevation to facilitate distant, unobstructed propagation of microwave, radio and/or television signals. Communication towers have a very high aspect ratio and are inherently unstable. Stability is achieved with cables, fastened near the top of the transmission tower that extend downward and outward. These cables are anchored into the ground providing a large footprint for the tower. Because stability is achieved with cables, the base of the tower is often pinned. For three dimensional structures, the equivalent of a pinned joint is a ball and socket joint. A ball and socket joint provides reaction forces R_x, R_y and R_z, but moments cannot develop. A schematic illustration of a communications tower with four cables anchored to the ground plane is shown in Fig. 9.2. The forces acting on a communications tower are due to:

- The dead weight of the tower and cables
- The pretension, if any, applied to the cables
- The weight of communication or other equipment installed on the tower
- Snow and/or ice accumulations
- Transverse loads due to winds[1]

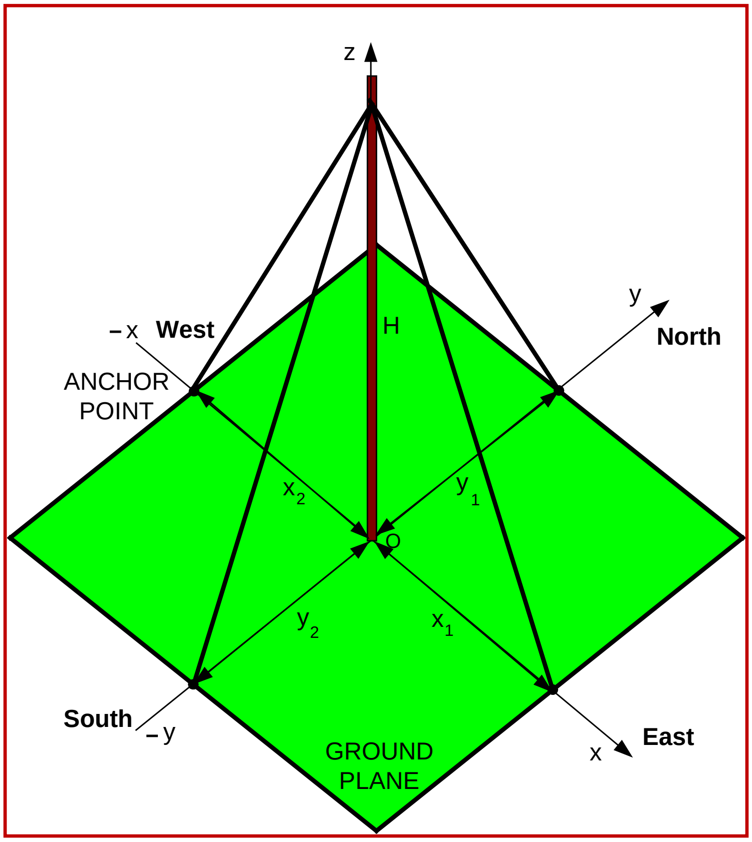


Fig. 9.2 Schematic illustration of a cable stabilized communications tower.

9.2 MODELING THREE DIMENSIONAL STRUCTURES

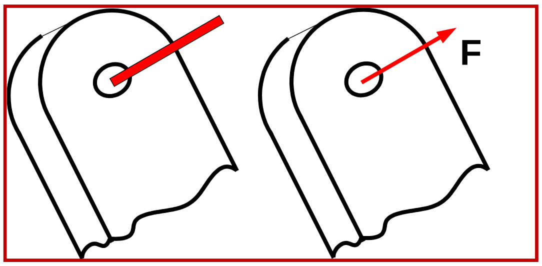
To perform an analysis of a three-dimensional structure, such as a communications tower, we follow the same six-step procedure used previously for trusses. The difference in the analysis involves the added complexity due to the inclusion of the third dimension. The additional dimension requires additional FBDs, and the application of all six of the equilibrium relations.

Space structures are always constrained by supports at several locations. These supports are modeled in FBDs with some combination of concentrated forces and moments that simulate the same

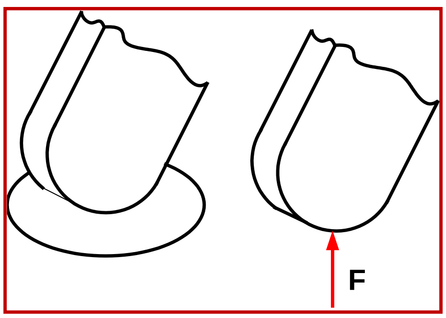
[1] In certain regions of the country, loading due to possible earthquakes should be included in this listing.

constraint provided by the supports. The forces and the moments produced by different types of supports that are common to space structures are listed in Table 9.1.

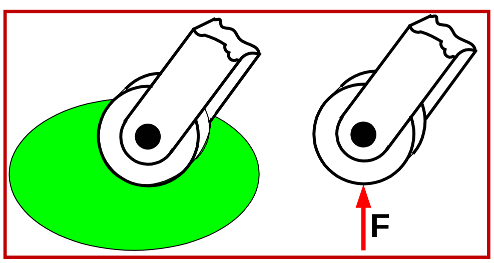
Table 9.1
Different types of supports or connections and their reactive forces and moments



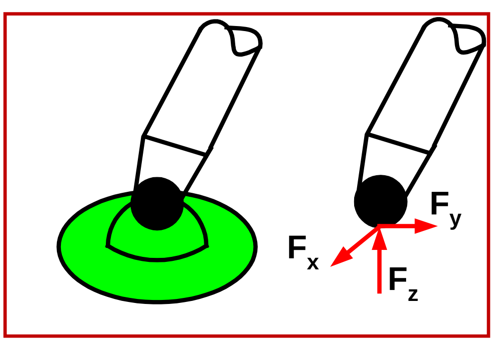
A. A cable connection is modeled with a tensile force acting in the direction of the cable and away from the structure.



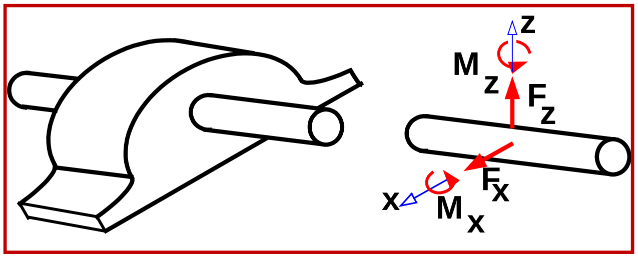
B. A rocker on a frictionless surface is modeled with a concentrated force perpendicular to the rocker's surface at the point of contact.



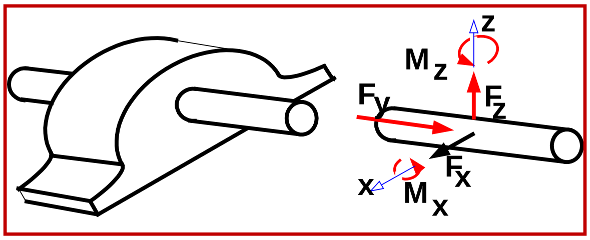
C. A roller on a flat surface is modeled with a concentrated force perpendicular to the roller's surface at its point of contact.



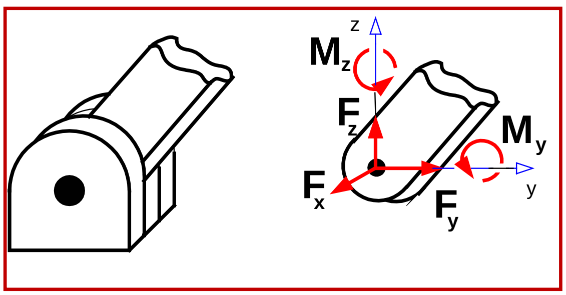
D. A ball and socket joint is modeled with three Cartesian force components. Moments cannot be supported.



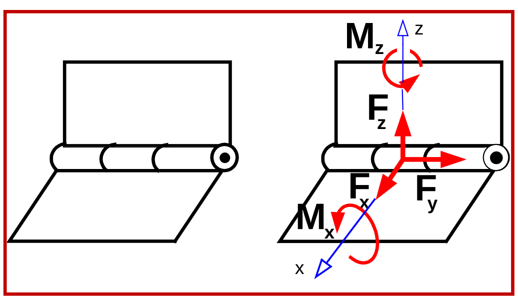
E. A journal bearing is represented with two forces perpendicular to the shaft and two moments about axes perpendicular to the shaft's axis. The shaft rotates freely about the y axis.



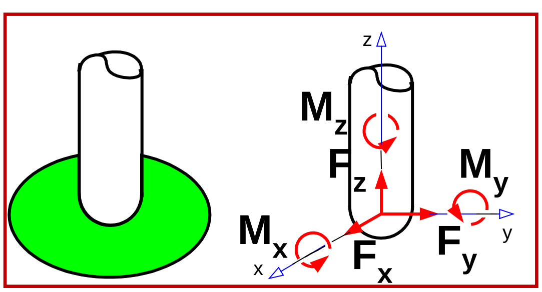
F. A thrust bearing is represented with three Cartesian forces and two moments about axes perpendicular to the shaft's axis. The shaft rotates freely about the y axis.



G. A pin and clevis connection is represented with three Cartesian forces and two moments about axes perpendicular to the pin's axis. The strut rotates freely about the x axis.



H. A hinge connection is represented with three Cartesian forces and two moments about axes perpendicular to the hinge pin's axis. The hinge rotates freely about the y axis.



I. A fixed support is modeled with six reactions (three Cartesian forces and three Cartesian moments).

When preparing models (FBDs) of three-dimensional structures, the supports of the structure are removed and replaced with the force and moment reactions, as described in Table 9.1[2]. In some instances, additional FBDs will be required to determine internal forces in select structural members. In many problems the structural members are classified as two force members, which enable you to apply an axial internal force with a direction that coincides with the orientation of the structural member.

9.3 THREE DIMENSIONAL SOLUTIONS

9.3.1 Three Dimensional Equilibrium Equations

When solving for forces in three-dimensional structures, the analysis becomes complex — often involving all six of the equations of equilibrium and several FBDs. Visualization of the structure and force components becomes more difficult. To alleviate the complexity and reduce visualization difficulties, we often employ vector mechanics, because expressing each force **F** or moment **M** in a complete vector notation reduces the equilibrium relations to two vector equations:

$$\Sigma \mathbf{F} = 0 \qquad \text{and} \qquad \Sigma \mathbf{M} = 0 \tag{9.1}$$

Recall that these vector equations can be expressed in terms of their Cartesian form as:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{9.1a}$$

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{9.1b}$$

9.3.2 Expressing Forces and Moments as Vectors

Moments are a product of a force times a distance. In most of our previous solutions, we were careful to define the distance as the perpendicular distance from the line of action of the force to the point about which the moments were determined. The definition of the moment does not change, but with the vector mechanics, we determine the moment in terms of a vector cross product.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{9.2}$$

where **r** is a position vector given by:

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} \tag{9.3}$$

The force vector is written as:

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \tag{9.4}$$

Recall, the properties of the cross vector product of the unit vectors and note:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = 0, & \mathbf{j} \times \mathbf{j} = 0, & \mathbf{k} \times \mathbf{k} = 0 \\ \mathbf{i} \times \mathbf{j} = -\,\mathbf{j} \times \mathbf{i} = \mathbf{k}, & \mathbf{j} \times \mathbf{k} = -\,\mathbf{k} \times \mathbf{j} = \mathbf{i}, & \mathbf{k} \times \mathbf{i} = -\,\mathbf{i} \times \mathbf{k} = \mathbf{j} \end{array} \tag{9.5}$$

---

[2] It is common practice to neglect moment reactions for structures supported by multiple bearings, pins, or hinges. In these cases, only the force reactions are necessary to keep the structure in equilibrium – the moment reactions are redundant (i.e. not required).

Combining Eqs. (9.2) to (9.5), we obtain:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k} \tag{9.6}$$

or

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k} \tag{9.7}$$

Comparing the results of Eqs. (9.6) with those of Eq. (9.7) yields:

$$\begin{aligned} M_x &= (r_y F_z - r_z F_y) \\ M_y &= (r_z F_x - r_x F_z) \\ M_z &= (r_x F_y - r_y F_x) \end{aligned} \tag{9.8}$$

When writing the vector cross product **r** × **F** to determine the moment **M,** we simultaneously write the equations for the moments M_x, M_y and M_z about the three Cartesian axes. We often employ the determinant to determine the moment vector about some origin, which is given by:

$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{9.9}$$

Let's apply these results to solve for the forces acting on the communications tower introduced previously. As we generate the solution to this example problem, a step-by-step procedure to follow in a design analysis of a three-dimensional structure is outlined.

EXAMPLE 9.1

For the communications tower described in Fig. 9.2, determine the safety factors[3] for the cable and the tower structure if:

- The dead weight of the tower is 20,000 lb.
- The receivers and transmitters mounted on the tower weigh 2,000 lb.
- The tower height H is 500 ft and the material used in fabrication has a yield strength of 35,000 psi.
- Each of the four cables has a diameter of 5/8 in. with a rated breaking strength of 41,200 lb. The length of the cables is adjusted so that they are snug, but they are not pretensioned.
- The anchor points for the tower are $x_1 = x_2 = y_1 = y_2 = 200$ ft.
- The maximum transverse load anticipated from a southwest wind is 1,600 lb. We assume that this load is applied at a distance z = 2H/3 from the ground plane.
- The anchor points are oriented so that the x-axis is due east and the y-axis is due north.
- The cross sectional area at the base of the tower is 11 in^2.
- A ball and socket joint supports the tower at point O.

[3] We will neglect bending in this analysis; however, the tower is subjected to bending stresses due to the transverse loads imposed by the wind.

Solution:

Step 1: Construct a FBD of the communications tower as shown in Fig. E9.1 .

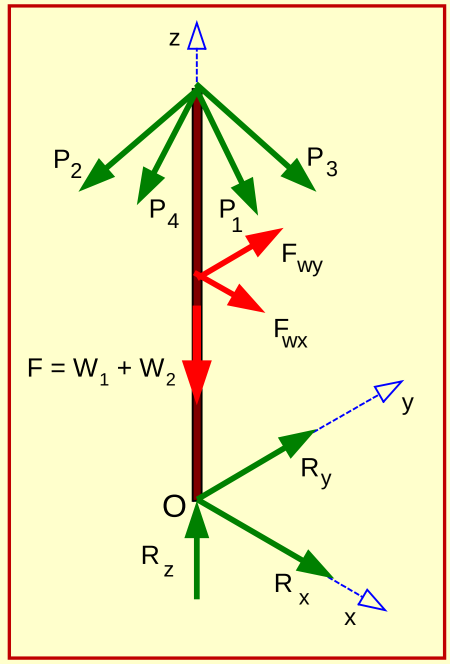


Fig. E9.1

Step 2: Count the forces acting on the structure and write the vector equation for each force. We have five forces acting on the tower including:

- Two-forces applied by the cables (two of the four cables are slack).
- The wind force.
- The reaction force at the base (origin) of the tower.
- The weight of the tower and transmission equipment.

Next let's prepare two-dimensional models (FBDs) showing the forces in the x-z and y-z planes as shown in Fig. E9.1a.

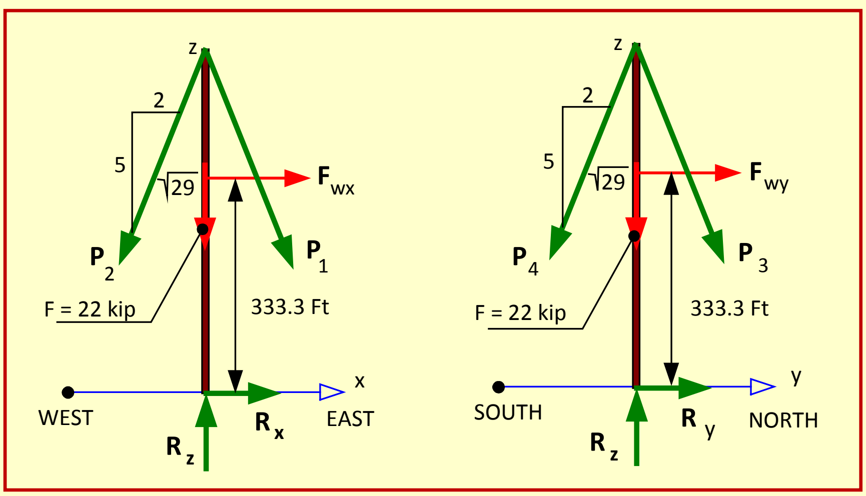
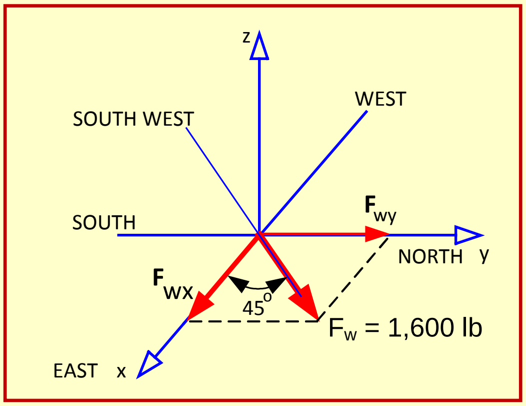


Fig. E9.1a

The FBD shown on the left is observed by standing to the south of the tower and looking north. Standing to the east and looking west, the FBD on the right is observed. By drawing two different views of the structure, we are able to show the components of the wind loading and the forces in all four of the cables. The FBDs are not complete, because we have not shown the magnitude of the wind forces F_{wx} and F_{wy}. Because the wind is from the southwest, we represent the wind force with the vector diagram, shown in Fig. E9.1b, and solve for the force components in the x and y directions.

Fig. E9.1b



The vector representing the force $\mathbf{P_2}$ on the cable anchored on the x-axis (west) is written as:

$$\mathbf{P_2} = P_2[(-2/\sqrt{29})\mathbf{i} + (-5/\sqrt{29})\mathbf{k}] \qquad \text{(a)}$$

Similarly, the force $\mathbf{P_4}$ on the cable anchored on the y-axis (south) is:

$$\mathbf{P_4} = P_4[(-2/\sqrt{29})\mathbf{j} + (-5/\sqrt{29})\mathbf{k}] \qquad \text{(b)}$$

Recall that $\mathbf{P_1} = \mathbf{P_3} = 0$, because these cables are slack under the specified wind conditions. Hence, these two cables do not support the tower. However, when the wind direction changes, these cables become functional and the vector equations must be changed to accommodate the new wind direction.

The wind force $\mathbf{F_w}$ is written as:

$$\mathbf{F_w} = F_w[\cos(45)\,\mathbf{i} + \sin(45)\,\mathbf{j}] = (\sqrt{2}/2)(1{,}600)[\mathbf{i} + \mathbf{j}] = (1131.4\,[\mathbf{i} + \mathbf{j}])\text{ lb} \qquad \text{(c)}$$

Note, the vector diagram, shown in Fig. E9.1b, is helpful in writing Eq. (c).

The reaction force **R** is written as:

$$\mathbf{R} = R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k} \qquad \text{(d)}$$

The weight of the tower structure (20 kip) and its equipment (2 kip) is expressed as:

$$\mathbf{F} = -(22 \times 10^3\,\mathbf{k})\text{ lb} \qquad \text{(e)}$$

We have described the five active forces acting on the communications tower in vector format.

Step 3: Write the equations for the position vectors for the five active forces. Let's consider the forces in order (i.e. 1, 2, …. 5).

For the two active cables that are attached at the top of the tower, we write the relations for the position vectors relative to the tower's base as:

$$\mathbf{r_1} = H\,\mathbf{k} \qquad \text{(f)}$$

$$\mathbf{r_2} = H\,\mathbf{k} \qquad \text{(g)}$$

For the wind force, the position vector is given by:

$$\mathbf{r_w} = (2/3)H\,\mathbf{k} \qquad \text{(h)}$$

Step 4: Employ the equilibrium relations to solve for the unknown forces.

First, consider the moments at the origin:

$$\Sigma\mathbf{M_O} = \Sigma\mathbf{r} \times \mathbf{F} = \mathbf{r_1} \times \mathbf{P_2} + \mathbf{r_2} \times \mathbf{P_4} + \mathbf{r_w} \times \mathbf{F_w} \qquad \text{(i)}$$

Note, the forces due to the reaction and dead weight forces don't cause a moment about the base because both the $\mathbf{r_5}$ and force vectors only have $\mathbf{k}$ components, and the cross product of any two vectors along the same direction is zero from Eqs. (9.5) and (9.6). Substituting the results for the forces and the position vectors in Eq. (9.9) yields three determinants:

$$\mathbf{M}_O = P_2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & H \\ \frac{-2}{\sqrt{29}} & 0 & \frac{-5}{\sqrt{29}} \end{vmatrix} + P_4\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & H \\ 0 & \frac{-2}{\sqrt{29}} & \frac{-5}{\sqrt{29}} \end{vmatrix} + \frac{\sqrt{2}}{2}F_w\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \frac{2H}{3} \\ 1 & 1 & 0 \end{vmatrix} = 0 \qquad \text{(j)}$$

Expanding these determinants gives:

$$\mathbf{M} = [(2/\sqrt{29})HP_4 - (\sqrt{2}/2)F_w(2H/3)]\,\mathbf{i} + [-(2/\sqrt{29})HP_2 + (\sqrt{2}/2)F_w(2H/3)]\,\mathbf{j} = 0 \qquad \text{(k)}$$

It is evident from Eq. (k) that:

$$\begin{aligned} M_x &= [(2/\sqrt{29})\,P_4 - (\sqrt{2}/2)F_w\,(2/3)] = 0 \\ M_y &= [-(2/\sqrt{29})\,P_2 + (\sqrt{2}/2)F_w\,(2/3)] = 0 \end{aligned} \qquad \text{(l)}$$

Recall that the magnitude of the wind load F_w = 1,600 lb, and solve Eqs. (l) to obtain:

$$P_2 = P_4 = 2{,}031 \text{ lb} \qquad \text{(m)}$$

Next, consider the forces:

$$\Sigma\mathbf{F} = \mathbf{P_2} + \mathbf{P_4} + \mathbf{F_w} + \mathbf{R} + \mathbf{F} = 0 \qquad \text{(n)}$$

Substituting Eqs. (a) through (e) into Eq. (n) yields:

$$\Sigma\mathbf{F} = P_2[(-2/\sqrt{29})\mathbf{i} + (-5/\sqrt{29})\mathbf{k}] + P_4[(-2/\sqrt{29})\mathbf{j} + (-5/\sqrt{29})\mathbf{k}]$$

$$+ (\sqrt{2}/2)\,F_w[\,\mathbf{i} + \mathbf{j}] + (R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k}) + (-22{,}000)\,\mathbf{k} = 0 \qquad \text{(o)}$$

We collect the coefficients of the unit vectors **i**, **j** and **k** and set each of these to zero. It is clear that the components of the sum of the forces must be zero in each of the three directions.

The coefficient of the unit vector **i** is:

$$\Sigma F_x = (-\,2/\sqrt{29})\,P_2 + (\sqrt{2}/2)\,F_w + R_x = 0 \qquad \text{(p)}$$

The coefficient of the unit vector **j** is:

$$\Sigma F_y = (-\,2/\sqrt{29})\,P_4 + (\sqrt{2}/2)\,F_w + R_y = 0 \qquad \text{(q)}$$

The coefficient of the unit vector **k** is:

$$\Sigma F_z = (-\,5/\sqrt{29})\,P_2 + (-\,5/\sqrt{29})\,P_4 + R_z - 22{,}000 = 0 \qquad \text{(r)}$$

From Eqs. (p) and (q), we show that:

$$R_x = R_y = -\,377.1 \text{ lb} \qquad \text{(s)}$$

The negative sign indicates that the direction of the forces R_x and R_y shown in the FBD should be reversed. Then from Eq. (r), we solve for R_z as:

$$R_z = (5/\sqrt{29})(P_2 + P_4) + 22{,}000$$

$$R_z = (10/\sqrt{29})(2{,}031) + 22{,}000 = 25{,}770 \text{ lb} \qquad \text{(t)}$$

Step 5: Determine the stresses at the base of the tower. At the base of the tower, the normal stress σ_z is given by:

$$\sigma_z = R_z/A = -\,(25{,}770)/(11) = -\,2{,}343 \text{ psi} \qquad \text{(u)}$$

The stress at the base of the tower is compressive.

It is not necessary to compute the stress in the cables, because the breaking strength is given in terms of the force applied to the cable, and we have already made this determination.

Step 5: Determine the safety factor and the margin of safety for the tower and the cables.

For the tower:

$$SF = S_y/\sigma = (35{,}000)/(2{,}343) = 14.94 \qquad MOS = SF - 1 = 13.94 \qquad \text{(v)}$$

For the cables:

$$SF = P_b/P = (41{,}200)/(2{,}031) = 20.29 \qquad MOS = SF - 1 = 19.29 \qquad \text{(w)}$$

Interpretation: The results indicate that the safety factors are very large. Develop arguments supporting the use of these large margins of safety in the design of a communications tower. Also, develop arguments for redesign with smaller diameter cables, smaller footprint and smaller cross sectional area for the tower structure.

EXAMPLE 9.2

For the derrick shown in Fig. E9.2, determine the margin of safety for the three supporting cables if the force F = 60 kN. The anchor points for cables AP and AQ lay in the x-y plane with coordinates P = (4, – 3, 0) m and Q = (– 4, – 3, 0) m. The derrick pole AO is 5 m high, and the boom BR is 6 m long. The derrick pole and boom are separate components. The boom is attached to the pole with a sleeve type bearing at point R that enables the boom to rotate about the pole. However, the boom is constrained along the z axis and cannot slide up or down on the pole. The cables are all 25 mm in diameter with a specified strength of 750 MPa. The pole is mounted on a ledge at point O with a ball and socket joint.

Solution:

Step 1: Let's consider the boom and pole of the derrick separately. First, prepare a FBD of the boom showing all of the unknown forces as shown in Fig. E9.2a. Examination of this FBD indicates that we have one cable force F_{BA}, the force F due to the load, and the reaction forces between the boom and the derrick pole. Note that no reactive moments are present at point R, only reactive forces.

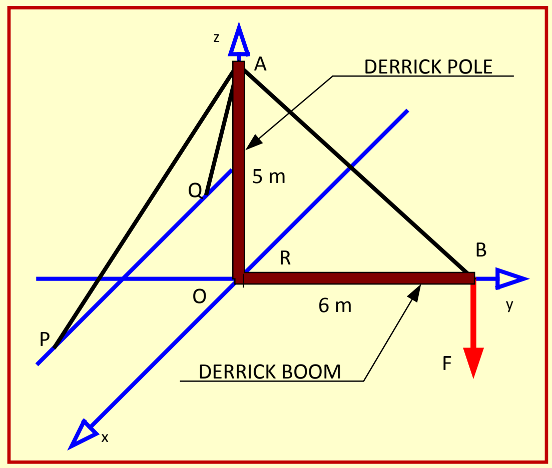


Fig. E9.2

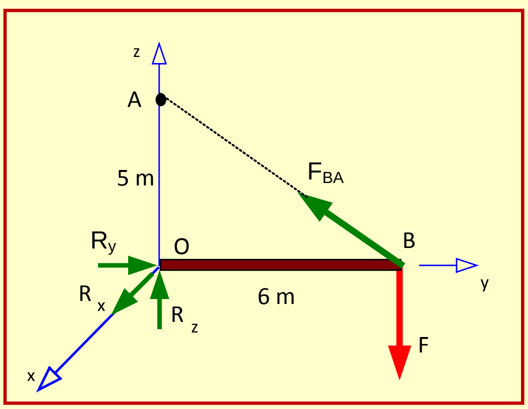


Fig. E9.2a

Step 2: Let's write the relations for the three forces in vector format.

$$\mathbf{F_{BA}} = F_{BA}\,\mathbf{u_{BA}} \qquad \mathbf{R} = R_x\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k} \qquad \mathbf{F} = -\,F\,\mathbf{k} = -\,(60\,\mathbf{k})\text{ kN} \quad \text{(a)}$$

Step 3: Next write the equation for the unit vector that provides the direction of the cable force F_{BA} and the position vector defining the location of point B relative to the origin:

$$\mathbf{u_{BA}} = (-\,6\,\mathbf{j} + 5\,\mathbf{k})/(36 + 25)^{1/2} = -\,0.7682\,\mathbf{j} + 0.6402\,\mathbf{k} \qquad \text{(b)}$$

$$\mathbf{r_B} = (6\,\mathbf{j})\text{ m} \qquad \text{(c)}$$

Step 4: Apply the equilibrium equations to the derrick boom.

$$\sum \mathbf{M_O} = \mathbf{r_B} \times \mathbf{F_{BA}} + \mathbf{r_B} \times \mathbf{F} = 0 \qquad \text{(d)}$$

$$\sum \mathbf{M_O} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ 0 & -0.7682 & 0.6402 \end{vmatrix} F_{BA} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ 0 & 0 & -60 \end{vmatrix} = 0$$

$$\sum \mathbf{M_O} = [(6)(0.6402)F_{BA} - (6)(60)]\,\mathbf{i} = 0$$

$$F_{BA} = 93.72 \text{ kN} \qquad \text{(e)}$$

$$\sum \mathbf{F} = \mathbf{F_{BA}} + \mathbf{R} + \mathbf{F} = 0$$

$$\sum \mathbf{F} = (R_x)\,\mathbf{i} + [(-\,0.7682)\,F_{BA} + R_y]\,\mathbf{j} + [(0.6402)\,F_{BA} + R_z - 60]\,\mathbf{k} = 0 \qquad \text{(f)}$$

Considering the coefficients of the **i**, **j** and **k** terms in Eq. (f) yields:

$$R_x = 0$$

$$R_z = 60 - 0.6402\,F_{BA} \qquad R_z = 60 - (0.6402)(93.72) = 0 \qquad \text{(g)}$$

$$R_y = 0.7682\,F_{BA} \qquad R_y = (0.7682)(93.72) = 72.0 \text{ kN} \qquad \text{(h)}$$

We observe that R_y acts along the axis of the boom tending to compress it. Having completed the analysis of the boom, we continue by considering the pole.

Step 5: Let's prepare a FBD of the pole showing all of the unknown forces as shown in Fig. E9.2b. Inspection of this FBD shows five unknown forces (F_{AP}, F_{AQ}, R_{xp}, R_{yp} and R_{zp}) and two known forces (F_{AB} and R_y). The reaction R_y is the boom pushing on the pole; whereas, the reactions R_{xp}, R_{yp} and R_{zp} are due to the interaction of the ball and socket joint at the base of the pole with the ledge supporting the derrick.

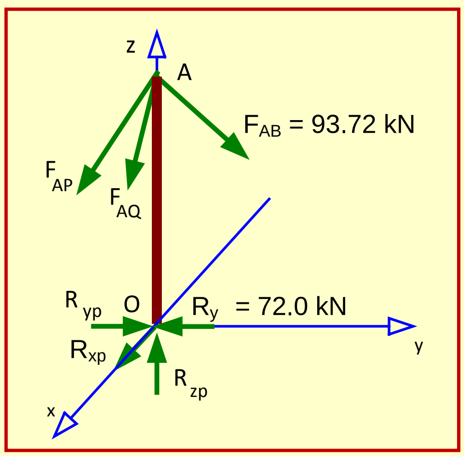


Fig. E9.2b

Step 6: Let's write the vector equations for the forces acting on the pole:

$$\mathbf{F_{AP}} = F_{AP}\,\mathbf{u_{AP}} \qquad \mathbf{F_{AB}} = (93.72\,\mathbf{u_{AB}}) \text{ kN}$$

$$\mathbf{F_{AQ}} = F_{AQ}\,\mathbf{u_{AQ}} \qquad \mathbf{R_p} = R_{xp}\,\mathbf{i} + R_{yp}\,\mathbf{j} + R_{zp}\,\mathbf{k} \tag{i}$$

Step 7: Next write the equations for the unit vectors and position vector that provide the directions of the cable forces and their location relative to the origin:

$$\mathbf{u_{AB}} = (6\,\mathbf{j} - 5\,\mathbf{k})/(36 + 25)^{1/2} = 0.7682\,\mathbf{j} - 0.6402\,\mathbf{k}$$

$$\mathbf{u_{AP}} = (4\,\mathbf{i} - 3\,\mathbf{j} - 5\,\mathbf{k})/(16 + 9 + 25)^{1/2} = 0.5657\,\mathbf{i} - 0.4243\,\mathbf{j} - 0.7071\,\mathbf{k} \tag{j}$$

$$\mathbf{u_{AQ}} = (-\,4\,\mathbf{i} - 3\,\mathbf{j} - 5\,\mathbf{k})/(16 + 9 + 25)^{1/2} = -\,0.5657\,\mathbf{i} - 0.4243\,\mathbf{j} - 0.7071\,\mathbf{k}$$

$$\mathbf{r_A} = (5\,\mathbf{k})\text{ m}$$

Step 8: Apply the equilibrium equations to the derrick pole.

$$\sum\mathbf{M_O} = \mathbf{r_A} \times \mathbf{F_{AB}} + \mathbf{r_A} \times \mathbf{F_{AP}} + \mathbf{r_A} \times \mathbf{F_{AQ}} = 0 \tag{k}$$

$$\sum M_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 0 & 0.7682 & -0.6402 \end{vmatrix} 93.72 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 0.5657 & -0.4243 & -0.7071 \end{vmatrix} F_{AP}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ -0.5657 & -0.4243 & -0.7071 \end{vmatrix} F_{AQ} = 0$$

Evaluation of this relation gives the coefficients of the unit vectors. For **i**, we obtain:

$$-\,(5)(0.7682)(93.72) + (5)(0.4243)\,F_{AP} + (5)(0.4243)\,F_{AQ} = 0 \tag{l}$$

For **j**, we obtain:

$$(5)(0.5657)\,F_{AP} - (5)(0.5657)\,F_{AQ} = 0 \tag{m}$$

Note that the coefficient of the unit vector **k** is zero.

From Eq. (m), it is clear that:

$$F_{AP} = F_{AQ} \tag{n}$$

Substituting Eq. (n) into Eq. (l) gives:

$$F_{AP} = F_{AQ} = [(0.7682)(93.72)]/[(2)(0.4243)] = 84.84\text{ kN} \tag{o}$$

This completes the solution for the forces in the cables. In a more complete analysis, we would employ $\sum\mathbf{F} = 0$ to determine the reactions at the base of the pole. It is suggested that you verify the results for $R_{xp} = 0$, $R_{yp} = 72.0$ kN and $R_{zp} = 180.0$ kN.

Step 9: Let's determine the stresses in each of the cables using the results for the forces given in Eqs. (e) and (o).

$$\sigma_{AB} = F_{AB}/A = (93.72 \times 10^3) /[\pi\ (12.5)^2] = 190.9 \text{ MPa}$$
$$\sigma_{AP} = \sigma_{AQ} = F_{AP}/A = (84.84 \times 10^3) /[\pi\ (12.5)^2] = 172.8 \text{ MPa} \quad \text{(p)}$$

Step 10: Finally, we determine the margin of safety for each of the cables.

For cable AB:

$$MOS = SF - 1 = S/\sigma_{AB} - 1 = (750)/(190.9) - 1 = 2.929 \quad \text{(q)}$$

For cables AP and AQ:

$$MOS = SF - 1 = S/\sigma_{AP} - 1 = 750/172.8 - 1 = 3.340 \quad \text{(r)}$$

Step 11: To complete the analysis, the results should be interpreted. We find, in this example, margins of safety of about 3 to 3.5 for the cables. The design is reasonably well balanced with stresses on the three cables nearly equal. The margin of safety is somewhat low for a derrick application, where large loads are lifted in the presence of workers.

EXAMPLE 9.3

The space truss, shown in Fig. E9.3, is supported at point A with a ball and socket joint and with three weightless links (two-force members) that are anchored at points B and D as shown in the illustration. A force **F** is applied to joint E. **F** is represented in vector format as:

$$\mathbf{F} = [40\ \mathbf{i} + 32\ \mathbf{j} - 60\ \mathbf{k}] \text{ kips}$$

Determine the internal forces in the three members (EB, EC and ED) that intersect at joint E. Also specify the diameter of the rods, if the truss is fabricated from 1020 HR steel. A safety factor of 6.0 based on yield strength is specified by the building code.

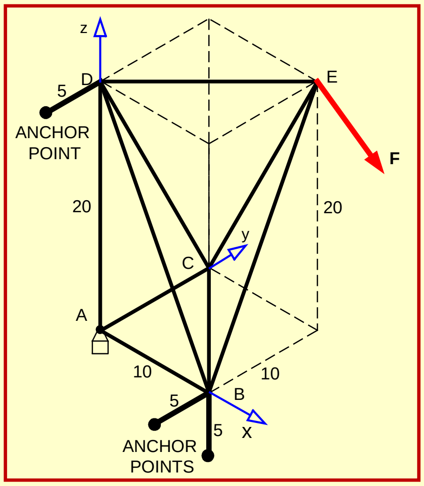


Fig. E9.3

Solution:

Step 1: Let's consider point E of the space truss and prepare a FBD showing the three unknown forces acting there, as shown in Fig. E9.3a. Examination of this figure indicates that we have a concurrent three-dimensional force system.

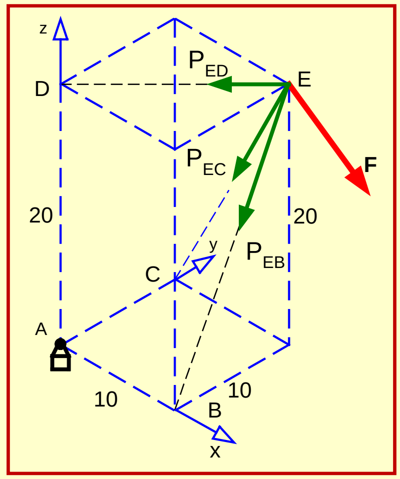


Fig. E9.3a

Step 2: Let's write the relations for the four forces in vector format.

$\mathbf{P_{EB}} = P_{EB}\,\mathbf{u_{EB}}$ $\qquad \mathbf{P_{EC}} = P_{EC}\,\mathbf{u_{EC}}$ $\qquad \mathbf{P_{ED}} = P_{ED}\,\mathbf{u_{ED}}$

$$\mathbf{F} = (40\,\mathbf{i} + 32\,\mathbf{j} - 60\,\mathbf{k}) \text{ kips} \qquad \text{(a)}$$

Step 3: Next write the equation for the unit vectors that gives the direction of the forces P_{EB}, P_{EC}, and P_{ED}.

$$\mathbf{u_{EB}} = (-10\,\mathbf{j} - 20\,\mathbf{k})/(100 + 400)^{1/2} = -0.4472\,\mathbf{j} - 0.8944\,\mathbf{k}$$

$$\mathbf{u_{EC}} = (-10\,\mathbf{i} - 20\,\mathbf{k})/(100 + 400)^{1/2} = -0.4472\,\mathbf{i} - 0.8944\,\mathbf{k} \qquad \text{(b)}$$

$$\mathbf{u_{ED}} = (-10\,\mathbf{i} - 10\,\mathbf{j})/(100 + 100)^{1/2} = -0.7071\,\mathbf{i} - 0.7071\,\mathbf{j}$$

Step 4: Apply the equilibrium equation to the force system acting at point E.

$$\Sigma\,\mathbf{F} = 0 \qquad \text{(c)}$$

Substituting Eqs. (a) and (b) into Eq. (c) yields:

$$\Sigma\,\mathbf{F} = \mathbf{i}\,(-0.4472P_{EC} - 0.7071\,P_{ED} + 40) + \mathbf{j}\,(-0.4472\,P_{EB} - 0.7071\,P_{ED} + 32)$$

$$+\,\mathbf{k}\,(-0.8944\,P_{EB} - 0.8944\,P_{EC} - 60) = 0 \qquad \text{(d)}$$

Setting each of the coefficients of the unit vectors to zero yields the three simultaneous equations given by:

$$-0.4472P_{EC} - 0.7071\,P_{ED} + 40 = 0$$

$$-0.4472\,P_{EB} - 0.7071\,P_{ED} + 32 = 0 \qquad \text{(e)}$$

$$-0.8944P_{EB} - 0.8944\,P_{EC} - 60 = 0$$

Solving Eq. (e) for the forces gives:

$$P_{EC} = -24.60 \text{ kip} \qquad P_{EB} = -42.48 \text{ kip} \qquad P_{ED} = 72.13 \text{ kip} \qquad \text{(f)}$$

Step 5: Determine the allowable design stress from the yield strength of the 1020 HR steel and the specified safety factor. From Appendix B-2 we find the yield strength S_y = 42 ksi.

$$\sigma_{design} = S_y / \mathbf{SF_y} = (42)/(6.0) = 7.00 \text{ ksi} \qquad \text{(g)}$$

Step 6: Solve for the cross sectional areas and diameters of the three members:

$$A = P/\sigma_{design} \qquad d = (4A/\pi)^{1/2} \qquad \text{(h)}$$

$$A_{EC} = (24.60)/(7.0) = 3.514 \text{ in.}^2 \qquad d = 2.115 \text{ in.} \qquad \Rightarrow d = 2\tfrac{1}{4} \text{ in.}$$

$$A_{EB} = (42.48)/(7.0) = 6.069 \text{ in.}^2 \qquad d = 2.780 \text{ in.} \qquad \Rightarrow d = 3 \text{ in.} \qquad \text{(i)}$$

$$A_{ED} = (72.13)/(7.0) = 10.30 \text{ in.}^2 \qquad d = 3.622 \text{ in.} \qquad \Rightarrow d = 3\tfrac{3}{4} \text{ in.}$$

Our steel supplier lists the standard sizes for 1020 HR steel rods from 1 to 3 in. in diameter as 1, 1 ¼, 1 ½, 1 ¾, 2, 2 ¼, 2 ½, 2 ¾, 3, 3 ¼, 3 ½, 3 ¾ and 4 in. You must round up diameters to the next closest standard size, to maintain a safety factor of at least 6.0 for each rod.

EXAMPLE 9.4

A tetrahedral space truss, shown in Fig. E9.4, supports a massive scoreboard and several sets of remotely controlled spotlights in an amphitheater. The base ABC of the space truss lies in the x-y plane (horizontal). The base triangle is connected to anchors in the roof beams by long cables at points A, B and C. Determine the size of the solid round rods required to fabricate members AB, AC and AD of the space truss. The safety factor for public arenas is 8, based on yield strength, and the material employed for all of the members is 1018A steel. The scoreboard and spotlights weigh 80 kN.

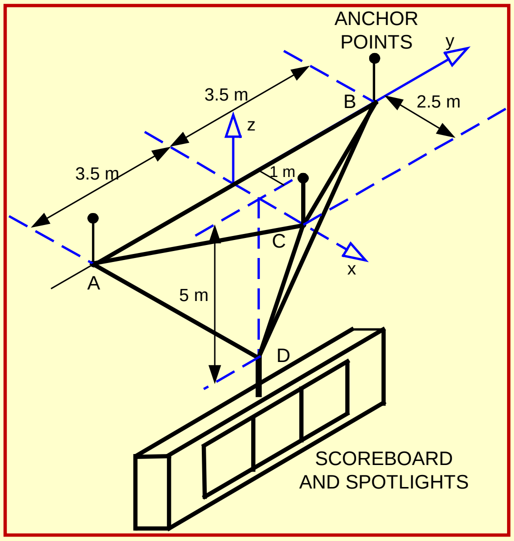


Fig. E9.4

Solution:

Step 1: Let's consider the space truss in its entirety and prepare a FBD showing the three unknown forces P_A, P_B and P_C acting on the anchor cables as shown in Fig. E9.4a.

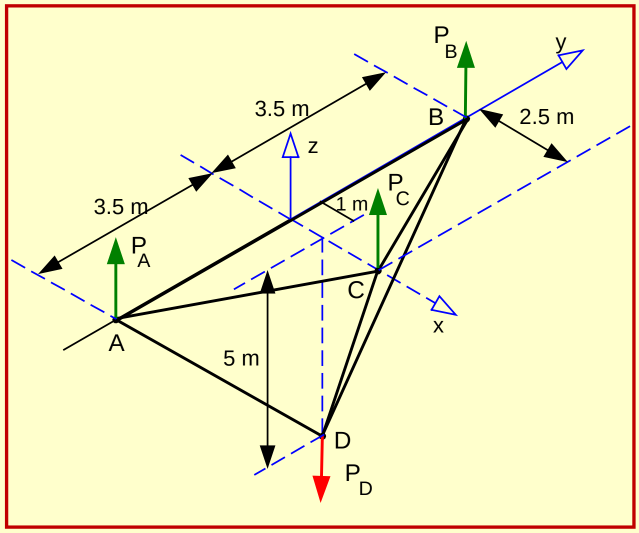


Fig. E9.4a

Step 2: Let's write the equation for the moments about point C in vector format and use the results to solve for P_A.

$$\Sigma \mathbf{M_C} = \mathbf{r_{CA}} \times \mathbf{P_A} + \mathbf{r_{CB}} \times \mathbf{P_B} + \mathbf{r_{CD}} \times \mathbf{P_D} = 0 \qquad \text{(a)}$$

The force vectors and position vectors are written as:

$$\mathbf{P_A} = P_A\,\mathbf{k} \qquad \mathbf{P_B} = P_B\,\mathbf{k} \qquad \mathbf{P_C} = P_C\,\mathbf{k} \qquad \mathbf{P_D} = -\,W\,\mathbf{k} \qquad \text{(b)}$$

$$\mathbf{r_{CA}} = -\,2.5\,\mathbf{i} - 3.5\,\mathbf{j} \qquad \mathbf{r_{CB}} = -\,2.5\,\mathbf{i} + 3.5\,\mathbf{j} \qquad \mathbf{r_{CD}} = -\,1.5\,\mathbf{i} - 5\,\mathbf{k} \qquad \text{(c)}$$

Substituting the results of Eqs. (b) and (c) into Eq. (a) yields:

$$\mathbf{M_C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2.5 & -3.5 & 0 \\ 0 & 0 & P_A \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2.5 & +3.5 & 0 \\ 0 & 0 & P_B \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.5 & 0 & -5 \\ 0 & 0 & -W \end{vmatrix} = \mathbf{0} \qquad \text{(d)}$$

Expanding the determinants gives two equations:

$$\mathbf{i}\,(-3.5\,P_A + 3.5\,P_B) = 0 \qquad \mathbf{j}\,(2.5\,P_A + 2.5\,P_B - 1.5\,W) = 0 \qquad \text{(e)}$$

Recall W = 80 kN and solve Eq. (e) to obtain:

$$P_A = P_B \qquad P_A = 24.0\ \text{kN} \qquad \text{(f)}$$

Step 3: Let's consider joint A from the space truss and prepare a FBD showing the three unknown forces P_{AB}, P_{AC}, and P_{AD}, as shown in Fig. E9.4b.

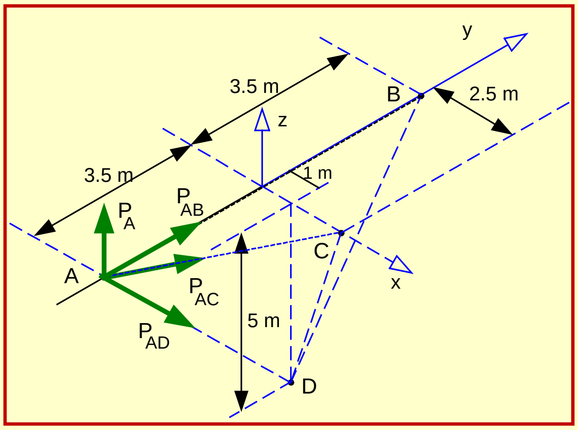


Fig. E9.4b

Step 4: Write the equilibrium equation for the **concurrent** force system acting on the pin at joint A as:

$$\Sigma\mathbf{F} = \mathbf{P}_{AB} + \mathbf{P}_{AC} + \mathbf{P}_{AD} + \mathbf{P}_{A} = 0 \qquad \text{(g)}$$

To employ this relation, we write the forces in vector format as:

$$\mathbf{P}_{AB} = P_{AB}\,\mathbf{u}_{AB} \qquad \mathbf{P}_{AC} = P_{AC}\,\mathbf{u}_{AC} \qquad \mathbf{P}_{AD} = P_{AD}\,\mathbf{u}_{AD} \qquad \text{(h)}$$

The unit vectors are given by:

$$\mathbf{u}_{AB} = \mathbf{r}_{AB}/r_{AB} = (7\mathbf{j})/(7) = \mathbf{j}$$

$$\mathbf{u}_{AC} = \mathbf{r}_{AC}/r_{AC} = (2.5\,\mathbf{i} + 3.5\mathbf{j})/[(2.5)^2 + (3.5)^2]^{1/2} = 0.5812\,\mathbf{i} + 0.8137\,\mathbf{j}$$

$$\mathbf{u}_{AD} = \mathbf{r}_{AD}/r_{AD} = (1.0\,\mathbf{i} + 3.5\mathbf{j} - 5\mathbf{k})/[(1.0)^2 + (3.5)^2 + (5)^2]^{1/2} \qquad \text{(i)}$$

$$\mathbf{u}_{AD} = 0.1617\,\mathbf{i} + 0.5659\,\mathbf{j} - 0.8085\,\mathbf{k}$$

Substituting the results of Eqs. (h) and (i) into Eq. (g) yields:

$$\mathbf{i}\,(0.5812\,P_{AC} + 0.1617\,P_{AD}) + \mathbf{j}\,(0.8137\,P_{AC} + 0.5659\,P_{AD} + P_{AB}) + \mathbf{k}\,(24.0 - 0.8085\,P_{AD}) = 0$$

Solving this equation gives:

$$P_{AD} = 29.68\text{ kN} \qquad P_{AC} = -8.257\text{ kN} \qquad P_{AB} = -10.08\text{ kN} \qquad \text{(j)}$$

Step 5: Determine the allowable design stress from the yield strength of the 1018A steel and the specified safety factor.

$$\sigma_{design} = S_y/\mathbf{SF_y} = (221)/(8) = 27.625\text{ MPa} \qquad \text{(k)}$$

Step 6: Solve for the cross sectional areas and diameters of the three members:

$$A = P/\sigma_{design} \qquad d = (4A/\pi)^{1/2} \qquad \text{(l)}$$

$$A_{AD} = (29.68 \times 10^3)/(27.625) = 1{,}074\text{ mm}^2 \qquad d_{AD} = 36.99\text{ mm} \Rightarrow d = 38\text{ mm}$$

$$A_{AC} = (8.257 \times 10^3)/(27.625) = 298.9\text{ mm}^2 \qquad d_{AC} = 19.51\text{ mm} \Rightarrow d = 20\text{ mm} \qquad \text{(m)}$$

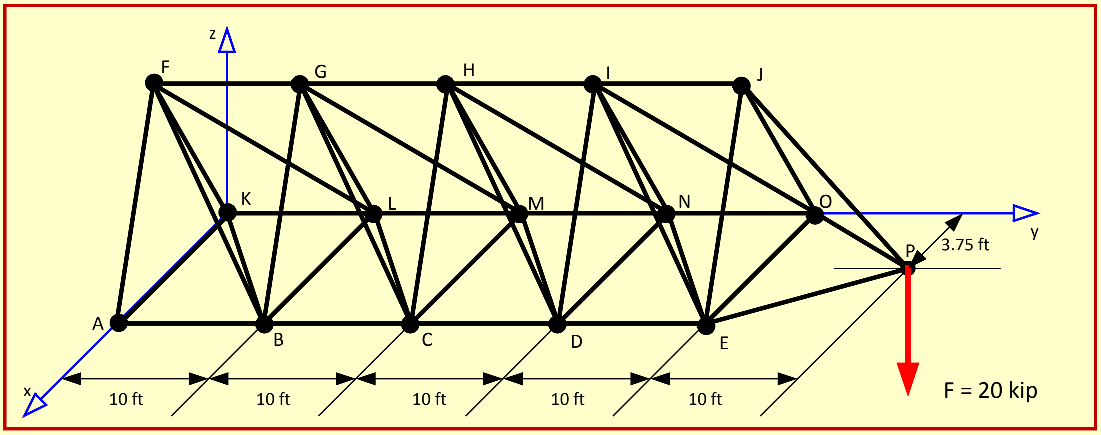
$A_{AB} = (10.08 \times 10^3)/(27.625) = 364.9\ mm^2$ $\qquad d_{AB} = 21.55\ mm \Rightarrow d = 22\ mm$

The diameters of the bars have been converted to the standard metric sizes available. A second analysis will be necessary to insure that the long thin rods used for members AC and AB, that are subjected to compressive forces, do not buckle.

EXAMPLE 9.5

The long horizontal boom of a construction crane is fabricated from a lattice of steel bars. A four-cell segment of the boom, presented in Fig. E9.5, shows that the lattice is periodic from points A to E with identical cells at 10 ft intervals. The rectangle AEOK lies in the x-y plane and the equilateral triangle AKF lies in the x-z plane. The lengths of the members AF = FK = AK = 7.5 ft. The crane is lifting a load F = 20 kip at point P. Determine the internal forces in members CD, MN, HI, HN, HD, and MD using the method of sections applied to this space structure.

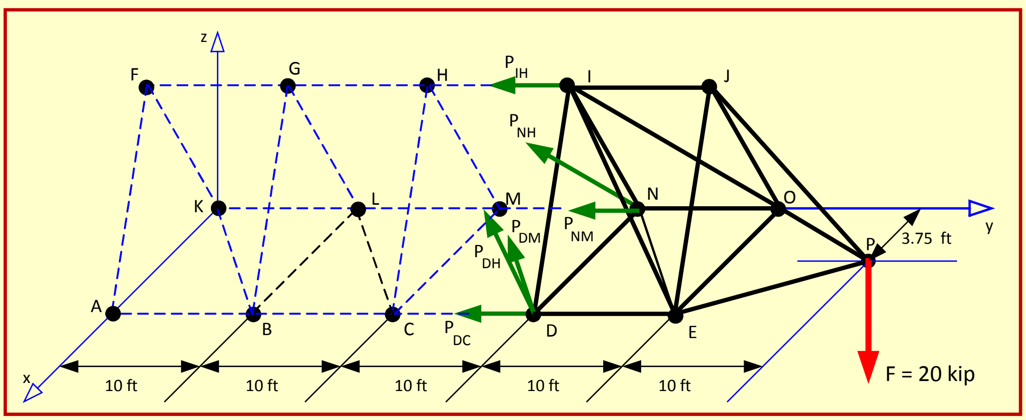
Fig. E9.5



Solution:

Step 1: Let's make a section cut perpendicular to the y-axis between points C and D and prepare a FBD of the right side of the boom. This FBD, presented in Fig. E9.5a, shows six unknown internal forces acting on the members CD, MN, HI, HN, DH, and DM.

Fig. E9.5a



Step 2: Let's write the equations for the moments about point D in vector format and use the results to obtain three equations in terms of the unknown internal forces P_{NM}, P_{NH} and P_{IH}.

$$\Sigma \mathbf{M_D} = \mathbf{r_{DN}} \times \mathbf{P_{NM}} + \mathbf{r_{DN}} \times \mathbf{P_{NH}} + \mathbf{r_{DI}} \times \mathbf{P_{IH}} + \mathbf{r_{DP}} \times \mathbf{F} = 0 \qquad \text{(a)}$$

The force vectors, unit vectors and position vectors in Eq. (a) are written as:

$$\mathbf{P_{NM}} = P_{NM}\,\mathbf{u_{NM}} \qquad \mathbf{P_{NH}} = P_{NH}\,\mathbf{u_{NH}} \qquad \mathbf{P_{IH}} = P_{IH}\,\mathbf{u_{IH}} \qquad \mathbf{F} = -F\,\mathbf{k} \qquad \text{(b)}$$

$$\mathbf{u_{NM}} = \mathbf{r_{NM}}/\,r_{NM} = -\,(10\,\mathbf{j})/(10) = -\,\mathbf{j}$$

$$\mathbf{u_{NH}} = \mathbf{r_{NH}}/\,r_{NH} = (+\,3.75\,\mathbf{i} - 10\,\mathbf{j} + 6.495\,\mathbf{k})/[(3.75)^2 + (10)^2 + (6.495)^2]^{1/2}$$

$$= 0.3\,\mathbf{i} - 0.8\,\mathbf{j} + 0.5196\,\mathbf{k} \qquad \text{(c)}$$

$$\mathbf{u_{IH}} = \mathbf{r_{IH}}/\,r_{IH} = -\,(10\,\mathbf{j})/(10) = -\,\mathbf{j}$$

Substituting Eq. (c) into Eq. (b) gives the internal forces in vector notation as:

$$\mathbf{P_{NM}} = -\,P_{NM}\,\mathbf{j}$$

$$\mathbf{P_{NH}} = P_{NH}\,(0.3\,\mathbf{i} - 0.8\,\mathbf{j} + 0.5196\,\mathbf{k}) \qquad \text{(d)}$$

$$\mathbf{P_{IH}} = -\,P_{IH}\,\mathbf{j} \qquad \mathbf{F} = -\,F\,\mathbf{k}$$

The position vectors are written as:

$$\mathbf{r_{DN}} = -\,7.5\,\mathbf{i} \qquad \mathbf{r_{DI}} = -\,3.75\,\mathbf{i} + 6.495\,\mathbf{k} \qquad \mathbf{r_{DP}} = -\,3.75\,\mathbf{i} + 20\,\mathbf{j} \qquad \text{(e)}$$

Substituting the results of Eqs. (d) and (e) into Eq. (a) yields:

$$\mathbf{M_D} = P_{NM}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7.5 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} + P_{NH}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7.5 & 0 & 0 \\ 0.3 & -0.8 & 0.5196 \end{vmatrix}$$

$$+\,P_{IH}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.75 & 0 & 6.495 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.75 & 20 & 0 \\ 0 & 0 & -F \end{vmatrix} = \mathbf{0} \qquad \text{(f)}$$

Expanding the determinants gives three equations:

$$\mathbf{i}\,(6.495\,P_{IH} - 20\,F) = 0$$

$$\mathbf{j}\,(3.897\,P_{NH} - 3.75\,F) = 0 \qquad \text{(g)}$$

$$\mathbf{k}\,(7.5\,P_{NM} + 6.0\,P_{NH} + 3.75\,P_{IH}) = 0$$

Recall F = 20 kip and solve Eqs. (g) to obtain:

P_{IH} = 61.59 kip $\quad$ P_{NH} = 19.25 kip $\quad$ P_{NM} = – 46.195 kip $\quad$ (h)

Step 3: Let's write the equation for equilibrium of the forces acting on the right side of the crane boom that is freed by the section cut:

$$\Sigma \mathbf{F} = \mathbf{P_{DC}} + \mathbf{P_{DH}} + \mathbf{P_{DM}} + \mathbf{P_{NM}} + \mathbf{P_{NH}} + \mathbf{P_{IH}} + \mathbf{F} = 0 \qquad \text{(i)}$$

Let's write each of these forces in vector notation as:

$$\mathbf{P_{DC}} = -P_{DC}\,\mathbf{j}$$

$$\mathbf{P_{DH}} = P_{DH}\,(-0.3\,\mathbf{i} - 0.8\,\mathbf{j} + 0.5196\,\mathbf{k})$$

$$\mathbf{P_{DM}} = P_{DM}\,(-0.6\,\mathbf{i} - 0.8\,\mathbf{j})$$

$$\mathbf{P_{NM}} = 46.195\,\mathbf{j} \qquad \text{(j)}$$

$$\mathbf{P_{NH}} = 5.775\,\mathbf{i} - 15.40\,\mathbf{j} + 10.0\,\mathbf{k}$$

$$\mathbf{P_{IH}} = -61.59\,\mathbf{j}$$

$$\mathbf{F} = -20.0\,\mathbf{k}$$

Substitute Eqs. (j) into Eq. (i) and collect all of the coefficients of **i, j** and **k** to obtain three equations containing the three unknown internal forces:

i: $\quad -0.3\,P_{DH} - 0.6\,P_{DM} + 5.775 = 0$

j: $\quad -P_{DC} - 0.8\,P_{DH} - 0.8\,P_{DM} + 46.195 - 15.40 - 61.59 = 0 \qquad$ (k)

k: $\quad 0.5196\,P_{DH} + 10.0 - 20.00 = 0$

Solving Eqs. (k) yields:

P_{DH} = 19.25 kip $\quad$ P_{DM} = 0 $\quad$ P_{DC} = – 46.195 kip $\quad$ (l)

We note that the forces $P_{DC} = P_{NM}$ = – 46.195 kip are compressive along the stringers in the x-y plane. The force P_{IH} = 61.59 kip along the top stringer is tensile as expected. Because the internal force in the diagonal member DM = 0, the space truss exhibits symmetric behavior with $P_{DC} = P_{NM}$ and $P_{DH} = P_{NH}$.

9.4 THREE DIMENSIONAL EQUILIBRIUM

The emphasis in the preceding sections of this chapter has been on the application of the equilibrium equations to solve for unknown forces in the members of space structures. However, applications of the equilibrium relations to other three-dimensional bodies such as machine components are often required in engineering studies. While the geometry of the bodies differs between structures and machine components, the approach is the same. You prepare FBDs, write appropriate equilibrium equations for each FBD and solve these equations for the unknown forces. To demonstrate these non-structural applications, consider two examples.

EXAMPLE 9.6

A local firm has constructed a small crane consisting of a boom supported by two steel wires BC and DE as shown in Fig. E9.6. The boom is fixed to the supporting wall with a ball and socket joint at point A. The wires are anchored into the wall at points C and E. Each wire has a diameter of 0.250 in. and an ultimate tensile strength of 180 ksi. Determine the maximum weight that can be supported by the boom, the support reactions at point A and the forces in both support wires.

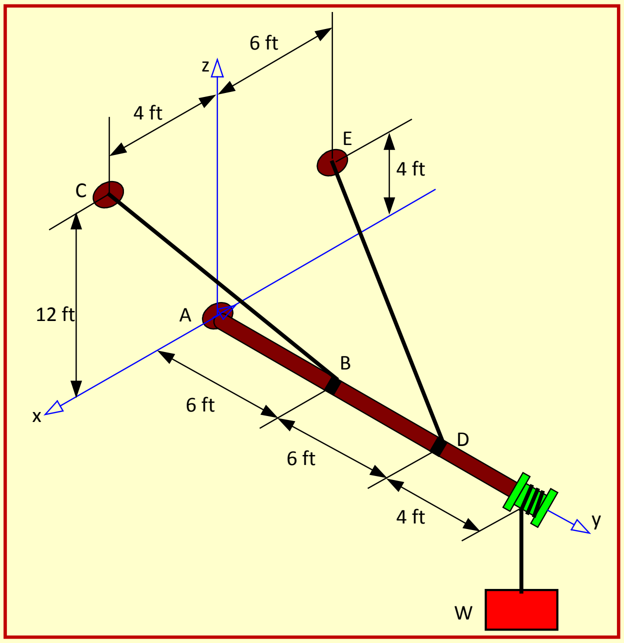


Fig. E9.6

Solution:

Step 1: Let's consider the crane in its entirety and prepare a FBD showing the three unknown forces at the ball and socket joint and the forces in the wires as shown in Fig. E9.6a.

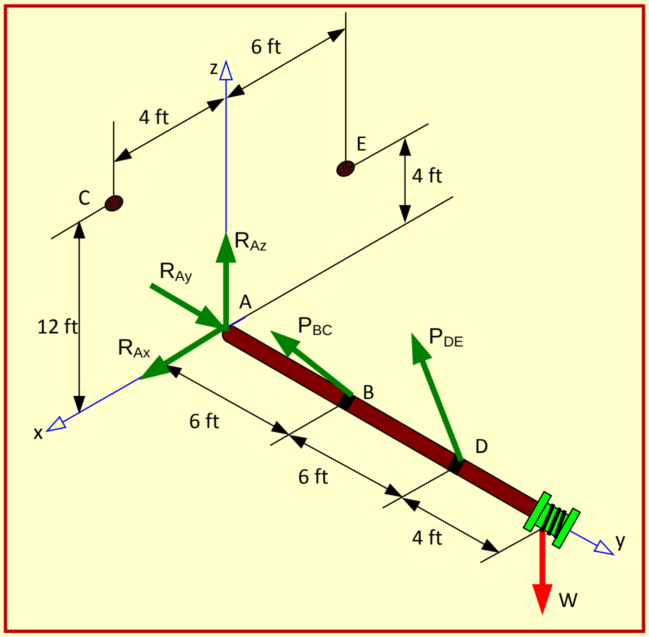


Fig. E9.6a

Step 2: Write the equations for the forces in the wires using a vector format.

$$\mathbf{P_{BC}} = P_{BC}\, \mathbf{u_{BC}} = P_{BC}\,(4\,\mathbf{i} - 6\,\mathbf{j} + 12\,\mathbf{k})/(14) \qquad \text{(a)}$$

$$\mathbf{P_{DE}} = P_{DE}\, \mathbf{u_{DE}} = P_{DE}\,(-6\,\mathbf{i} - 12\,\mathbf{j} + 4\,\mathbf{k})/(14) \qquad \text{(b)}$$

Step 3: Consider equilibrium of the structure. From the FBD in Fig. E9.6a, we write the six equations of equilibrium as:

$$\Sigma F_x = R_{Ax} + (4/14)\,P_{BC} - (6/14)\,P_{DE} = 0 \qquad \text{(c)}$$

$$\Sigma F_y = R_{Ay} - (6/14)\,P_{BC} - (12/14)\,P_{DE} = 0 \qquad \text{(d)}$$

$$\Sigma F_z = R_{Az} + (12/14)\,P_{BC} + (4/14)\,P_{DE} - W = 0 \qquad \text{(e)}$$

$$\Sigma M_x = (6)(12/14)P_{BC} + (12)(4/14)P_{DE} - (16)W = 0 \qquad \text{(f)}$$

$$\Sigma M_y = 0 \qquad \text{(g)}$$

$$\Sigma M_z = -(6)(4/14)P_{BC} + (12)(6/14)P_{DE} = 0 \qquad \text{(h)}$$

Step 4: Solve Eq. (h) and determine the maximum tension allowable in either wire to obtain:

$$P_{BC} = 3P_{DE} \qquad \text{(i)}$$

$$P_{Max} = S_u\, A = (180 \times 10^3)[(\pi/4)(0.25)^2] = 8{,}836 \text{ lb} \qquad \text{(j)}$$

Step 5: Use Eqs. (h) and (i) to solve for the forces in the wires. Because $P_{BC} > P_{DE}$ it is clear that:

$$P_{BC} = P_{Max} = 8{,}836 \text{ lb} \quad \text{and} \quad P_{DE} = 2{,}945 \text{ lb} \qquad \text{(k)}$$

Step 6: Solve for the maximum weight that can be lifted by substituting Eq. (k) into Eq. (f):

$$W_{Max} = (1/16)[(72/14)(8{,}836) + (48/14)(2945)] = 3{,}471 \text{ lb} \qquad \text{(l)}$$

Step 7: Solve for the reaction forces at point A by substituting Eqs. (k) and (l) into Eqs. (c), (d) and (e) to obtain:

$$R_{Ax} = -\,1{,}262 \text{ lb} \qquad R_{Ay} = 6{,}311 \text{ lb} \qquad R_{Az} = -\,4{,}944 \text{ lb} \qquad \text{(m)}$$

EXAMPLE 9.7

A hand operated lifting mechanism called a windless utilizes a crank to rotate a drum, as shown in Fig. E9.7. The shaft of the mechanism is supported by a thrust bearing at point A and a journal bearing at point B. The handle of the crank in the position shown is in the y-z plane. For this position of the crank handle, determine the force P required to lift a weight W = 150 lb. Also determine the support reactions at the thrust and journal bearings. Note that moment reactions can be neglected at both bearings.

Solution:

Step 1: Consider the windless in its entirety and prepare a FBD showing the three unknown forces at the thrust bearing, the two forces at the journal bearing and the applied forces as shown in Fig. E9.7a.

Step 2: Employ vector mechanics in this solution by writing the equation for the moment vector about point A as:

$$\Sigma \mathbf{M_A} = \mathbf{r_W} \times \mathbf{F_W} + \mathbf{r_{AB}} \times \mathbf{F_B} + \mathbf{r_F} \times \mathbf{F} = 0 \qquad \text{(a)}$$

The forces are given by:

$$\mathbf{F_W} = -\,(150\,\mathbf{k}) \text{ lb} \qquad \mathbf{F_B} = F_{Bx}\,\mathbf{i} + F_{Bz}\,\mathbf{k} \qquad \mathbf{F} = F\,\mathbf{i} \qquad \text{(b)}$$

The position vectors are given by:

$$\mathbf{r_W} = (0.5\,\mathbf{i} + 2\,\mathbf{j}) \text{ ft} \qquad \mathbf{r_{AB}} = (4\,\mathbf{j}) \text{ ft} \qquad \mathbf{r_F} = (5.5\,\mathbf{j} - 1.4\,\mathbf{k}) \text{ ft} \qquad \text{(c)}$$

Step 3: Substitute the results from Eqs. (a) and (b) into Eq. (9.9) and solve to obtain:

$$\Sigma M = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.5 & 2 & 0 \\ 0 & 0 & -150 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 0 \\ F_{Bx} & 0 & F_{Bz} \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5.5 & -1.4 \\ F & 0 & 0 \end{vmatrix} = 0 \qquad \text{(d)}$$

$$(-300\,\mathbf{i} + 75\,\mathbf{j}) + (4\,F_{Bz}\,\mathbf{i} - 4\,F_{Bx}\,\mathbf{k}) + F\,(-\,1.4\,\mathbf{j} - 5.5\,\mathbf{k}) = 0 \qquad \text{(e)}$$

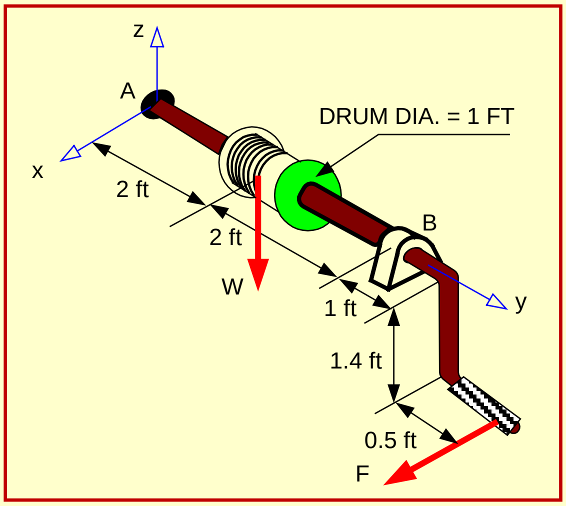


Fig. E9.7

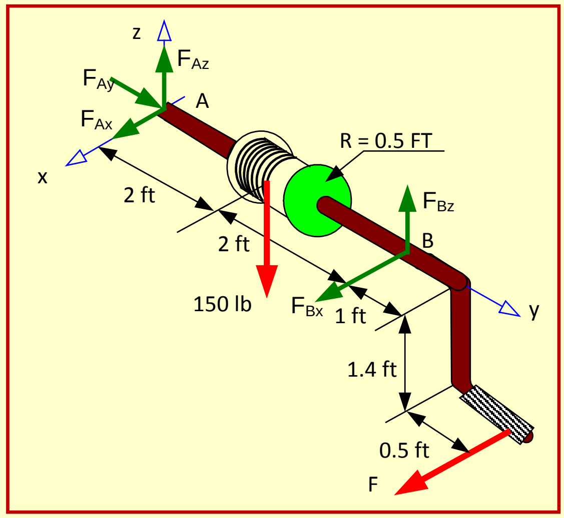


Fig. E9.7a

Rearrange the terms in Eq. (e) and write:

$$(-\ 300 + 4\ F_{Bz})\ \mathbf{i} + (75 - 1.4\ F)\ \mathbf{j} + (-\ 4\ F_{Bx} - 5.5\ F)\ \mathbf{k} = 0 \qquad \text{(f)}$$

Equate each coefficient in Eq. (f) to zero to obtain:

$$(-\ 300 + 4\ F_{Bz}) = 0 \qquad (75 - 1.4\ F) = 0 \qquad (-\ 4\ F_{Bx} - 5.5\ F) = 0 \qquad \text{(g)}$$

Solve Eqs. (g) for the unknowns F, F_{Bx} and F_{Bz} to obtain:

$$F = 53.57\text{ lb} \qquad F_{Bz} = 75.0\text{ lb} \qquad F_{Bx} = -\ 73.66\text{ lb} \qquad \text{(h)}$$

Step 4: To determine the forces at the thrust bearing, let's use Σ**F** = 0 and write:

$$\Sigma F_x = F_{Ax} + F_{Bx} + F = 0 \qquad \Sigma F_y = F_{Ay} = 0 \qquad \Sigma F_z = F_{Az} + F_{Bz} - 150 = 0 \qquad \text{(i)}$$

Substituting the results from Eqs. (h) into Eqs (i) yields:

$$F_{Ax} = 20.09\text{ lb} \qquad F_{Ay} = 0 \qquad F_{Az} = 75.0\text{ lb} \qquad \text{(j)}$$

9.5 INTERNAL FORCES AND MOMENTS

We have solved for reaction forces and moments in several examples. The reactions, moments or forces are external to the structure or the machine; however, they create stresses in the members of the structure. For example, the members in a space structure are uniaxial, two-force members. The external forces F produce an internal force P, as we showed previously. But how is the internal force generated? Applying an axial force on a uniaxial member produces a tensile stresses σ that is uniformly distributed over the cross section of the axial member. We cannot see the stresses. Indeed, we represent the stresses by making a section cut on the uniaxial member a draw a cluster of arrows representing the stresses. If the stresses σ are integrated over the area A of the cross section, we generate an internal force or an internal moment.

We can generate six internal forces or moments, which include: P_z, V_x, V_y, M_x, M_y, and M_z. An illustration of these internal forces is presented in Fig. 9.3. In this figure, the tube like structure is loaded at its end with external forces, shown as red arrows. The vertical segment of the tube has been cut and we have drawn a coordinate system at the location of the section cut. We have also replaced the effect of the bottom section of the vertical tube with internal forces, P_z, V_x, V_y, M_x, M_y, and M_z. These internal forces and moments are represented with green arrows. Often we use the symbol T For torque instead of M_z and F_z instead of P_z.

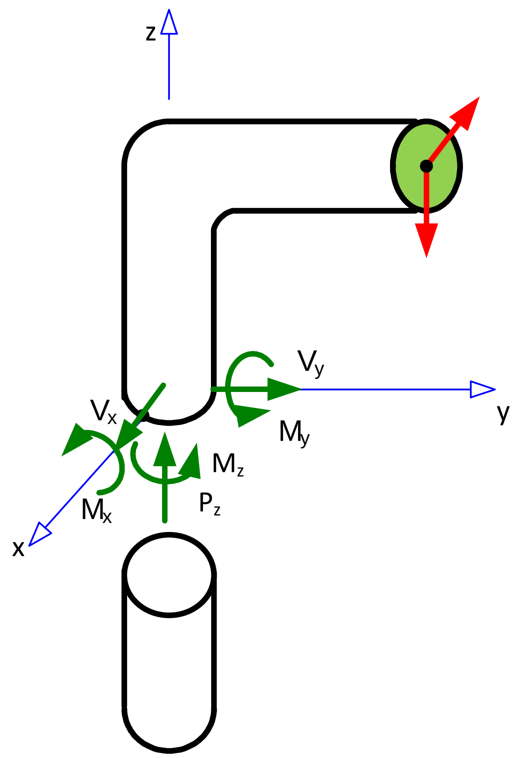


Fig. 9.3 Internal forces and moments revealed with a section cut on the vertical portion of the tube structure.

After preparing a FBD that shows internal and external forces and moments, a coordinate system and significant dimensions, the task is to determine the internal forces. To demonstrate this process, let's consider two examples.

EXAMPLE 9.8

A traffic sign along a city street is cantilevered from a pole, as illustrated in Fig. E9.8. The pressure due to the wind impinging on the sign is uniformly distributed over its area and equal to 14 lb/ft^2 and oriented in the x direction. The pole is fabricated from a tube with a 14 in. outside diameter and a 0.20 in. wall thickness. Determine the internal forces and the moments acting at the base of the pole.

Solution:

Begin by preparing a FBD of the sign and pole, as shown in Fig. 9.8a. Notice that the coordinate system is shown together with the three internal forces and three internal moments at the section cut. The effective force due to the wind pressure acts at the center of the sign and is given by:

$$F_x = pA = (14)(6)(12) = 1{,}008 \text{ lb} \qquad \text{(a)}$$

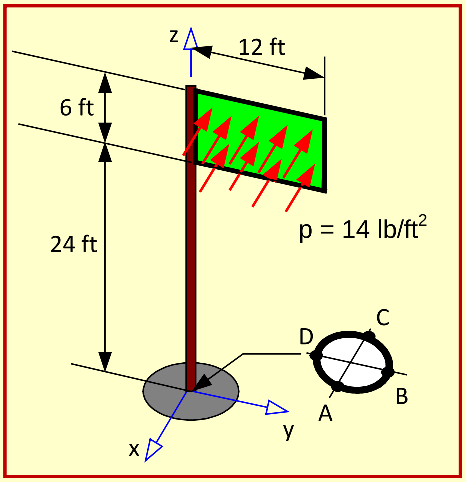
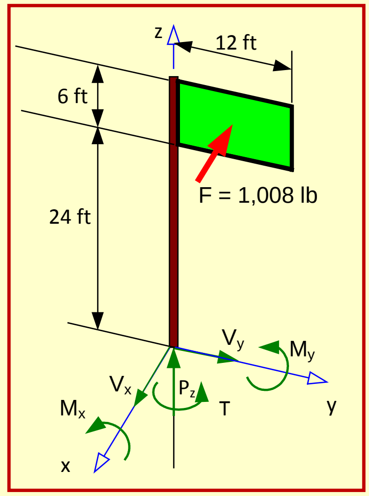


Fig. E9.8 A traffic sign along a city street is cantilevered from a pole

Fig. E9.8a A FBD showing the wind pressure acting on the sign.



Next we write the six equations of equilibrium to obtain:

$\Sigma F_x = 0$	$V_x = 1{,}008$ lb	(b)
$\Sigma F_y = 0$	$V_y = 0$	(c)
$\Sigma F_z = 0$	$P_z = 0$	(d)
$\Sigma M_x = 0$	$M_x = 0$	(e)
$\Sigma M_y = 0$	$M_y = (1{,}008)(27) = 27{,}220$ ft-lb	(f)
$\Sigma M_z = 0$	$T = -(1{,}008)(6) = -6{,}048$ ft-lb	(g)

Finally we prepare a drawing of the base where internal forces T, M_y and V_x act, as shown in Fig E9.8b. Note that M_x, P_z and V_y are not shown because they vanish.

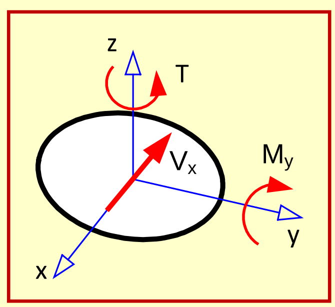


Fig. E9.8b

EXAMPLE 9.9

A machine component is fabricated from a tube that has been formed into a 90° bend, as illustrated in Fig. E9.9. The outside diameter of the tube is 60 mm and its inside diameter is 50 mm. A design engineer wants to place an access hole in the tube at a location 150 mm from its base and is concerned with the possibility of failure by doing so. He assigns you the task of determining the internal forces and moments at this elevation on the tube.

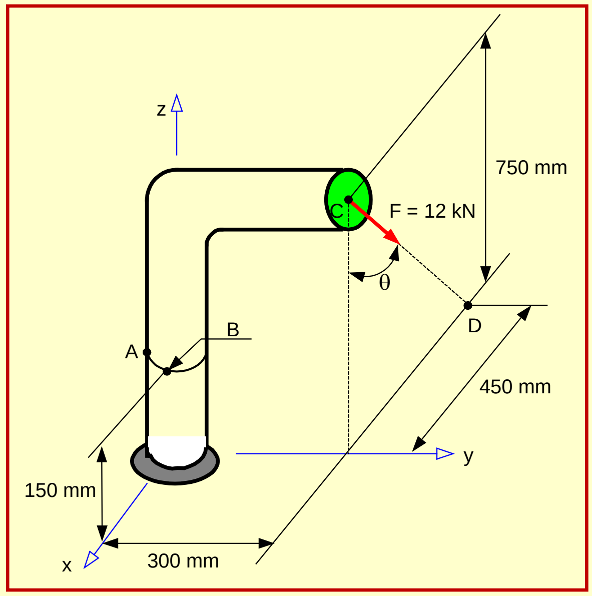


Fig. E9.9

Solution:

Let's first determine the angle θ and resolve the force **F** into its components F_x and F_z.

$$\theta = \tan^{-1}(450/750) = 30.96° \quad \text{(a)}$$

$$F_x = F \sin\theta = 12 \sin(30.96°) = 6.173 \text{ kN} \quad \text{(b)}$$

$$F_z = F \cos\theta = 12 \cos(30.96°) = 10.29\text{kN} \quad \text{(c)}$$

Next prepare a FBD of the segment of the tube with a section cut at the 150 mm elevation, as shown in Fig. E9.9a.

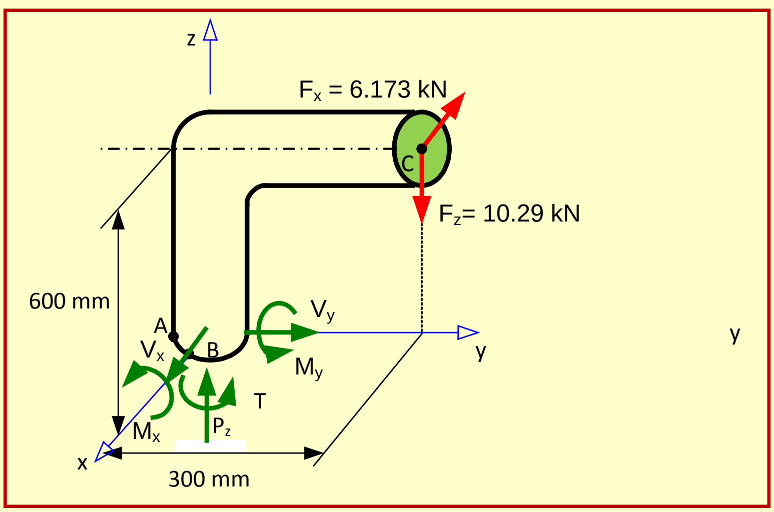


Fig. E9.9a

Let's use this FBD together with the equilibrium relations to determine the internal forces at the section cut.

$\Sigma F_x = 0$	$V_x = 6.173$ kN	(d)
$\Sigma F_y = 0$	$V_y = 0$	(e)
$\Sigma F_z = 0$	$P_z = 10.29$ kN	(f)
$\Sigma M_x = 0$	$M_x = (10.29 \times 10^3)(0.300) = 3{,}087$ N-m	(g)
$\Sigma M_y = 0$	$M_y = (6.173 \times 10^3)(0.600) = 3{,}704$ N-m	(h)
$\Sigma M_z = 0$	$T = -(6.173 \times 10^3)(0.300) = -1{,}852$ N-m	(i)

In this example we observe that only V = 0 and all the remaining five internal forces are required to maintain the segment of the structure in equilibrium.

9.6 SUMMARY

When solving for forces in members of three-dimensional structures, there are two different mathematical approaches. One approach employs trigonometric methods for determining force and moment components in the three directions. Another approach employs vector analysis. The first approach is intuitively obvious and the second is more mathematically elegant. In both approaches, we model the structure with FBDs. When modeling, we suggest the use of two or three-view FBDs to facilitate the visualization process.

When using the vector approach, the equilibrium relations can be simply expressed as two vector equations:

$$\Sigma \mathbf{F} = 0 \quad \text{and} \quad \Sigma \mathbf{M} = 0 \tag{9.1}$$

However for the trigonometric approach, we must employ six equilibrium relations in the solution of the unknown internal and external forces:

$$\Sigma F_x = 0; \quad \Sigma F_y = 0; \quad \Sigma F_z = 0 \tag{9.1a}$$

$$\Sigma M_x = 0; \quad \Sigma M_y = 0; \quad \Sigma M_z = 0 \tag{9.1b}$$

With the vector analysis approach, we write vector relations defining all of the forces (**F**) acting on the structure as well as the position vectors (**r**) that define the location of the forces relative to a point about which the moments act. Both types of vectors are expressed in component form as:

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} \tag{9.3}$$

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \tag{9.4}$$

Then, the moments are determined from the force vectors and the position vectors by employing the vector cross product.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k} \qquad (9.6)$$

For the trigonometric approach, moments are determined by using Eq. (1.12) and following the procedure described in Chapter 2. Note that this analysis must be performed three times to analyze moments about the x-, y-, and z-axes.

Finally, the equilibrium relations given by either Eq. (9.1) or Eqs. (9.1a) and (9.1b) are used to solve for the unknown internal and external forces acting on the three-dimensional structure.

Three-dimensional structures are difficult to analyze in comparison to two-dimensional structures. In both cases, the approach is the same — model the structure by constructing the appropriate FBDs and apply the equilibrium relations. The difficulty in analyzing three-dimensional structures arises for two reasons. First, visualization is often a problem for some students. Second, many forces are usually involved and the analysis becomes long and involved. The need for visualization is mitigated to some degree by employing a vector mechanics approach, where the process leads one to understand the geometry and dimensions of the body under consideration. Dividing the solution into different steps and completing each step as an individual analysis addresses the difficulties associated with the problem's length. This stepwise approach was demonstrated in several examples.

Finally, the technique used to solve for internal forces and moments was discussed and examples were presented to demonstrate the solution process.

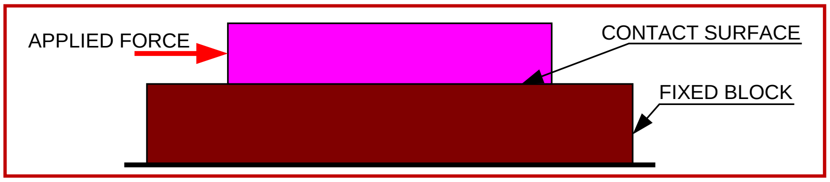
CHAPTER 10

FRICTION

10.1 INTRODUCTION

Friction occurs when two bodies are in contact and forces are applied to one or both bodies. Frictional forces develop at the contacting surfaces that tend to inhibit the sliding of one body relative to the other. For example, it is friction between the tires and road that keeps an automobile from sliding when it travels around a curve. This ability of a tire to maintain contact with the road is often referred to as **traction**. We depict this situation, with the illustration given in Fig. 10.1.

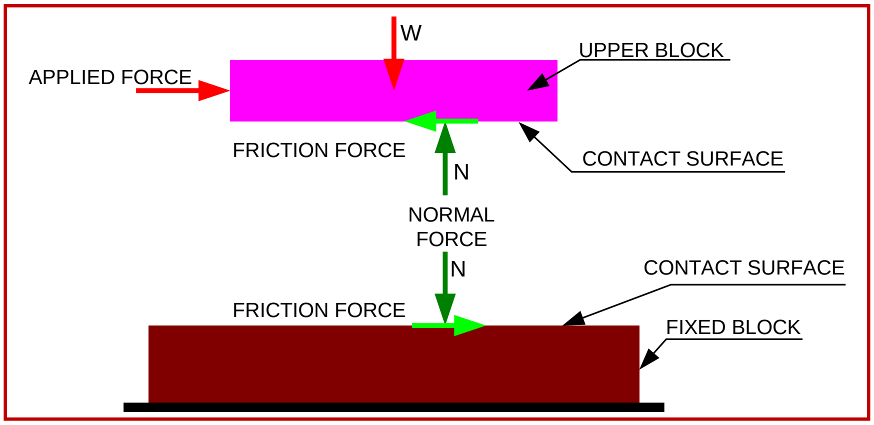
Fig. 10.1 Two bodies in contact with a force applied to the upper block.



If we slowly increase the lateral force applied to the upper block, we observe that it initially remains stationary. The block remains in equilibrium even with an applied force of significant magnitude. Is this behavior consistent with Newton's second law, which states $\Sigma\mathbf{F} = m\mathbf{a}$? Let's look more closely at the upper block in Fig. 10.1, and prepare a FBD showing **all** the forces that act on this body. The FBD for both the upper and lower blocks is shown in Fig. 10.2. Because the upper block is in equilibrium, it is clear that the frictional force must act in the opposite direction to the applied force. The frictional force on the fixed (lower) block is the same magnitude, but it is in the opposite direction, because of the law of action and reaction. Normal forces **N** also develop between the two contacting surfaces. The normal forces are perpendicular to the contacting surfaces, equal in magnitude and opposite in direction.

If we continue to increase the applied force, the upper block eventually will begin to slide over the surface of the lower block, indicating some upper limit on the frictional forces. Before we explore this upper limit, let's consider several factors that influence friction. We know that friction only occurs with two or more bodies in contact, so it is reasonable to examine in detail the two contact surfaces. Consider a magnified view of the surfaces presented in Fig. 10.3.

Fig. 10.2 Free body diagram showing the friction forces developed on both bodies at the contact surface.



It is evident in Fig. 10.3 that smooth surfaces are not really smooth when viewed at high magnification. Instead, the surfaces are made up of valleys and peaks, which we call **asperities**. Contact occurs only when two opposing asperities are aligned. Because the alignment of two opposing peaks is rare, the actual area of contact A_c is small compared to the area A of the entire surface. For reasonably smooth surfaces, the ratio of A_c /A is:

$$A_c / A \leq 0.001 \tag{10.1}$$

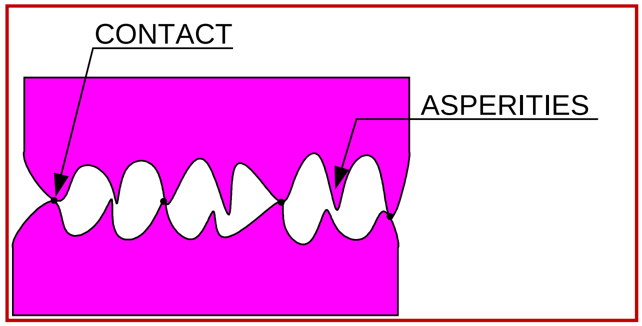


Fig. 10.3 Schematic illustration of contacting surfaces.

When a lateral force is applied to one of the bodies a resistive force (friction) develops, because some of the asperities are in contact. In some cases, these asperities mechanically interfere with each other, and must be sheared off before motion between the surfaces may occur. In other cases, the contacting asperities bond together, and this bond must be fractured before sliding can occur. The shearing of asperities and the fracture of the contact bonds produce wear of the sliding surfaces.

In the study of mechanics, we ignore the microscopic characteristics of the surfaces and treat friction from a more pragmatic viewpoint. However, we recognize the difference between dry and fluid friction. In **dry** or **Coulomb**[1] **friction** the asperities are in contact, and appreciable friction forces develop to resist sliding motion between the contacting surfaces. In **fluid friction**, we introduce a lubricant that separates the two surfaces, as illustrated in Fig. 10.4.

When a lubricant is present the frictional forces are reduced significantly. This is the reason for using oil to lubricate the engines in automobiles. The resistance is no longer due to shearing (fracturing) the asperities to initiate sliding. Instead, much smaller forces are required to shear the lubricant. In this chapter, we will be concerned with only **dry or Coulomb friction**. Lubrication theory is a topic introduced later in the Mechanical Engineering curriculum.

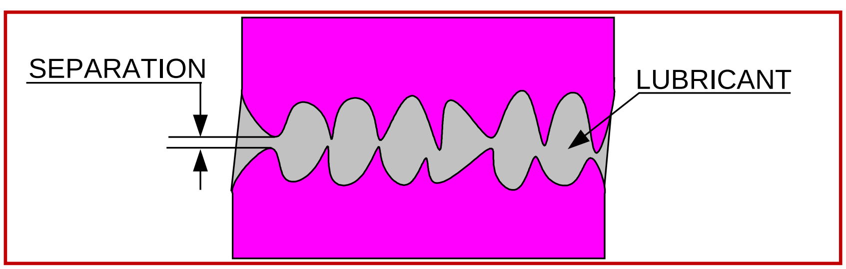


Fig. 10.4 The lubricant separates the surfaces preventing contact of the asperities.

[1] Charles Augustine Coulomb, a military engineer, developed the general theory of friction and published his work in a lengthy memoir — Théorie des Machines Simple, En ayant égard au frottement de leurs parties, et a la roideur des Cordages, Mémoires de Mathématique et de Physique, vol. X, 1785.

10.2 STATIC AND DYNAMIC FRICTION

Let's return to the model, depicted in Fig. 10.1 with two blocks in contact, and examine the conditions required to maintain the upper block in equilibrium. The body is two-dimensional and three equations of equilibrium must be satisfied, namely $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M_O = 0$. The FBD of the upper block, presented in Fig. 10.5, shows the applied lateral force F, the weight of the block W, the vertical reaction force N due to the supporting block and the friction force F_f. The equilibrium relations yield:

$$\Sigma F_y = 0 \Rightarrow N = W \qquad \text{(a)}$$

$$\Sigma F_x = 0 \Rightarrow F = F_f \qquad \text{(b)}$$

The normal force N is located along the contact surface at position s, to satisfy the moment equation of equilibrium.

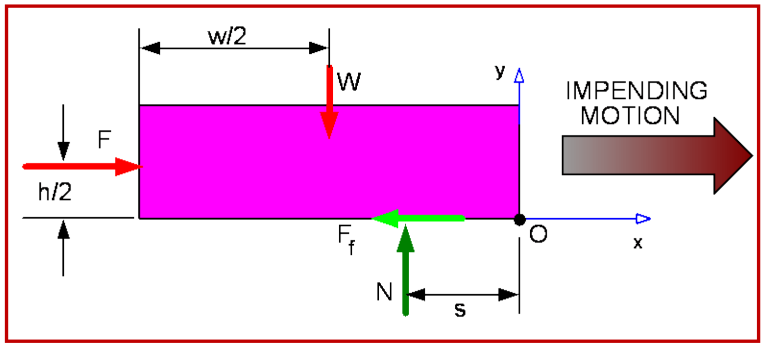


Fig. 10.5 FBD of a block prior to the initiation of sliding.

From the relation $\Sigma M_O = 0$ and Eq. (a), we may determine the location s for the normal force N as:

$$s = (W\,w - F\,h)/(2W) \qquad (10.2)$$

Note s = w/2 in the absence of a lateral force F. When F increases the value of s decreases, as the normal force shifts to the right. When F = (wW)/h, s = 0, and the block is on the verge of tipping. We will consider whether the block slides or tips later in this chapter.

The magnitude of the frictional force is variable. The frictional force becomes as large as necessary to prevent sliding up to some maximum force, when the asperities break free. The maximum frictional force is approximated by:

$$F_f \leq \mu N \qquad \text{before motion}$$

$$(F_f)_{Max} = \mu N \qquad \text{at impending motion} \qquad (10.3)$$

where μ is the coefficient of friction.

The coefficient of friction depends on the materials in contact, and to a significant degree on the surface finish and cleanliness of the contact surfaces. A listing of the coefficient of friction μ for a number of different materials is given in Table 10.1; however, it is important to note the very wide variations in μ for all material combinations. The variation is so large that it is advisable to conduct simple experiments to measure the coefficient of friction of the materials involved in a design of a machine component.

The maximum frictional force increases linearly with the normal force N and is considered to be independent of the area of the surfaces in contact providing that A_c/A remains very small, as indicated in Eq. (10.1). When the normal force becomes extremely large, the asperities undergo plastic deformation and the ratio A_c/A increases markedly. Another situation leading to significant increases in A_c/A is when one of the two bodies is fabricated from a soft material, such as rubber or plastic. In these two cases, the area of the surfaces becomes important, and Eq. (10.3) is no longer valid.

Table 10.1
Approximate coefficient of friction for contact between different materials

Surface Materials	Coefficient of Friction μ (static)
Metal on metal	0.15 – 0.50
Metal on stone or concrete	0.30 – 0.70
Metal on leather	0.30 – 0.60
Metal on wood	0.20 – 0.60
Wood on wood	0.30 – 0.70
Wood on stone or concrete	0.30 – 0.70
Rubber on concrete	0.60 – 0.90
Metal on brake pad	0.20 – 0.40
Metal on Ice	0.03 – 0.05

For ordinary conditions Eq. (10.3) is valid, and the friction force increases linearly with the applied lateral force F, as shown in Fig. 10.6, until motion (sliding or tipping) occurs.

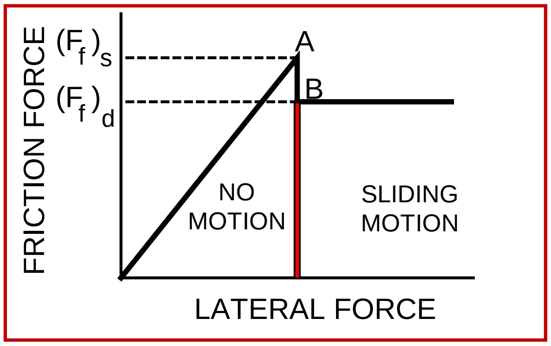


Fig. 10.6 Graph of frictional force with the applied lateral force.

The results presented in Fig. 10.6 show that the frictional force increases with the applied lateral force until sliding initiates, when the asperities begin to fail. At this instant, the frictional force decreases abruptly (from points A to B in Fig. 10.6). After the onset of sliding, the frictional force remains constant at a value of $(F_f)_d$ even with further increases in the lateral force. The friction force that develops under sliding conditions is smaller than the friction under static conditions. The mechanical interference of the asperities still occurs, but the time required for adhesion (bonding) of the asperities is not sufficient for bonding, when one surface is sliding relative over the other. Under sliding conditions, the kinetic friction force, $(F_f)_d$ is approximated by:

$$(F_f)_d = \mu_d N \qquad (10.4)$$

where μ_d is the dynamic or kinetic coefficient of friction, which is usually approximated by multiplying the static value by 0.75 to 0.80.

10.3 MEASURING THE COEFFICIENT OF FRICTION

The coefficient of friction μ exhibits significant variation due to affinity of different surfaces, surface roughness and the cleanliness of the two mating surfaces. For these reasons, it is advisable to measure the coefficient of friction of the materials to be employed in the design of machine components and structures. The measurement of μ is relatively easy to accomplish, using the simple experimental arrangement shown in Fig. 10.7.

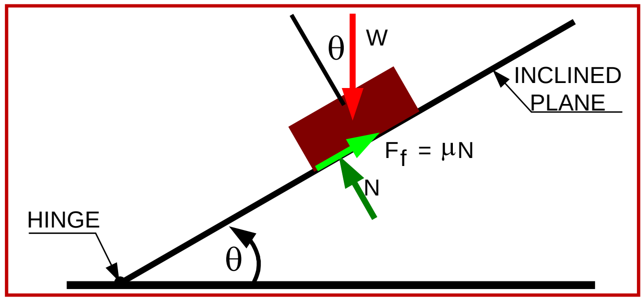


Fig. 10.7 Illustration of a method for measuring the static coefficient of friction.

A block of the first material is placed on an inclined plane with a surface fabricated from the second material. The angle of inclination of the plane θ is slowly increased until the block begins to slide. The angle required to initiate movement (sliding) is known as the angle of impending motion θ_s. This angle is measured with a protractor. Let's perform an equilibrium analysis to show the relation between μ and θ_s.

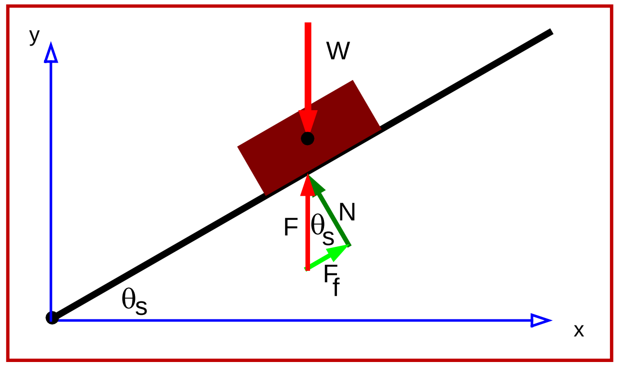


Fig. 10.8 FBD of the block on the inclined plane at the position of impending motion.

We begin the equilibrium analysis by drawing a FBD of the block, where the inclined plane is at the position of impending motion as shown in Fig. 10.8. We have rearranged the forces acting on the block to show the resultant force **F,** which is given by:

$$\mathbf{F} = \mathbf{N} + \mathbf{F_f} \tag{a}$$

Next, consider the equilibrium relation $\Sigma F_y = 0$, and write:

$$\Sigma F_y = F - W = 0 \quad \Rightarrow \quad F = W \tag{b}$$

We also note that:

$$N = F \cos \theta_s = W \cos \theta_s \tag{c}$$

$$(F_f)_{Max} = F \sin \theta_s = W \sin \theta_s \tag{d}$$

Recall Eq. (10.3) and use Eqs. (c) and (d) to write:

$$\mu = (F_f)_{Max}/N = \sin \theta_s / \cos \theta_s = \tan \theta_s \tag{10.5}$$

Clearly, we can measure the coefficient of friction quickly and efficiently with this simple experiment. The angle θ_s is related to the coefficient of friction by the tangent function. The angle θ_s is sometimes referred to as the angle of repose, which we will show is equal to the angle of friction[2] φ.

[2] The angle of friction φ is the maximum angle that a block resting on a ramp can remain at rest prior to sliding occurring.

Let's consider a block resting on the horizontal surface. Apply a lateral force F to the block until it is just ready to slide (**impending motion**). The FBD, for the block under these loading conditions, is illustrated in Fig. 10.9.

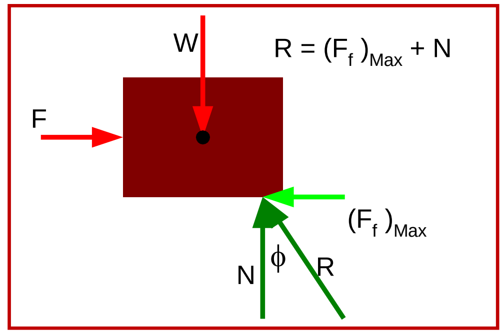


Fig. 10.9 FBD of a block subjected to applied forces associated with impending motion.

To analyze this FBD note that the angle of friction is defined as:

$$\tan \phi = (F_f)_{Max} / N \qquad \text{(a)}$$

But with the block subject to **impending motion**, we may write $(F_f)_{Max} = \mu N$. Substituting this relation into Eq. (a) leads to:

$$\tan \phi = \mu \qquad (10.6)$$

Combine Eqs. (10.5) and (10.6) to yield:

$$\theta_s = \phi \qquad (10.7)$$

From these simple derivations, we have established that the angle of repose θ_s is equal to the angle of friction ϕ.

EXAMPLE 10.1

A sales representative states that the coefficient of friction μ for a flat neoprene-fabric belt is 0.66 when the belt is operated against a smooth-metal pulley. You obtain a sample of the belt, and plan to conduct an inclined plane experiment to measure the coefficient of friction μ. As a part of the plan for your experiment, determine the expected angle of repose.

Solution:

Recall Eq. (10.5) and write:

$$\theta_s = \tan^{-1} \mu = \tan^{-1} (0.66)$$

$$\theta_s = 33.42° \qquad \text{(a)}$$

Your plan for the friction experiment should permit the inclined plane to tilt to at least 40°, and the protractor for measuring the angle of repose should be accurate to 0.5°.

EXAMPLE 10.2

In conducting repeated experiments with samples of the flat neoprene-fabric belt, you record five different values of the angle θ_s including 34.2°, 32.5°, 33.6°, 34.1°, and 31.2°. Determine the average value of the friction coefficient μ and its range.

Solution:

Recall Eq. (10.5) and write:

$$\mu = \tan \theta_s \qquad \text{(a)}$$

The results from Eq. (a) are given in the table below:

θ_s	μ
34.2°	0.6796
32.5°	0.6371
33.6°	0.6644
34.1°	0.6771
31.2°	0.6056

The average value of μ is $\Sigma\mu/n = (3.2638)/(5) = 0.6528$. (b)

The range in the data for μ is $\mu_{max} - \mu_{min} = 0.6796 - 0.6056 = 0.0740$ (c)

EXAMPLE 10.3

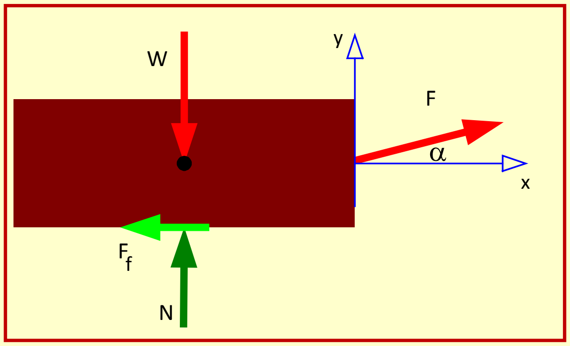
You are attending a county fair and decide to watch an exhibition involving a harnessed team of mules that are pulling a sled loaded with concrete blocks. The load is pulled only short distances (about two meters). You are amazed at the number of concrete blocks that are placed on the sled, and decide to conduct an analysis to determine the force F with which the team pulls. Draw the FBD necessary to conduct the analysis, and list the information necessary to determine the force F.

Solution:

Let's represent the sled, its weight, and the weight of the concrete blocks with a single rectangular block as shown in Fig. E10.3. This FBD represents a model of the mule team, the sled and its load. The mule team pulls with a force F that is oriented at an angle α relative to the ground. The friction force F_f provides the resistance in the x direction.

To determine the force F, we must gather information defining the angle α, the friction coefficient and the weight W. Suppose the sled together with the added concrete blocks weighs a total of 4,200 lb. The harness on the mules is shortened as much as possible to maximize the angle α at a value of 20°. The steel runners on the sled interact with a flat, bare, dry, clay field. If we assume that the coefficient of friction of metal (steel) on dry clay is 0.44, we have collected all of the information needed to determine F. Let's solve for the force F that the team of mules provides.

Fig. E10.3



Begin by writing the equilibrium relations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, which yield:

$$\Sigma F_x = F \cos \alpha - F_f = 0 \qquad (a)$$

$$\Sigma F_y = F \sin \alpha + N - W = 0 \qquad (b)$$

Then recall Eq. (10.3) and note the maximum load pulled by the mule team occurs when motion of the sled is **impending**. In this instance, the friction force is a maximum and we may write:

$$(F_f)_{Max} = \mu N \qquad (c)$$

Substituting Eq. (10.3) into Eq. (a) yields:

$$F = \mu N/\cos \alpha \qquad (d)$$

Substituting Eq. (d) into Eq. (b), and solving for the unknown force N leads to:

$$\mu N \tan \alpha + N - W = 0$$

$$N = W/(1 + \mu \tan \alpha) \qquad (10.8)$$

Finally, substitute Eq. (10.8) into Eq. (d) to obtain:

$$F = \frac{\mu W}{\cos\alpha(1+\mu\tan\alpha)} \qquad (10.9)$$

We have numbered the relations for the two unknown forces N and F in the sled example as Eqs. (10.8) and (10.9). These relations may be used in the solution of N and F for any block that is moved by sliding, providing μ, W, and α are known. Let's complete the solution and determine the capability of our mule team. Substituting into Eq. (10.9) gives:

$$F = \frac{(0.44)(4{,}200)}{\cos(20^\circ)[1+(0.44)\tan(20^\circ)]} = 1{,}695 \text{ lb} \qquad (e)$$

EXAMPLE 10.4

Consider a wooden crate that is resting on a level concrete floor of a warehouse. A worker attempts to slide the crate a short distance, as shown in Fig. E10.4. If the crate weighs 800 N and the coefficient of friction between the wooden crate and the concrete floor is μ = 0.44, determine the force that must be applied to slide the crate. Note that the crate is short and the worker must lean over to apply a force at an angle of 28° relative to the floor.

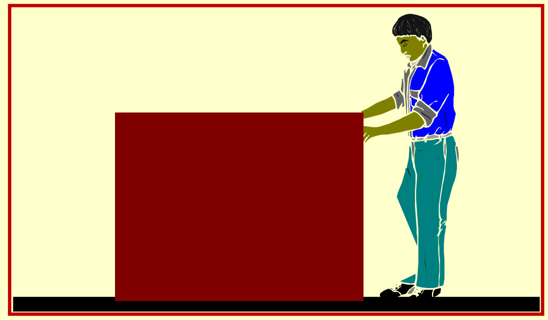


Fig. E10.4

Solution:

Begin the solution by constructing a FBD to model the physical situation as shown in Fig E10.4a.

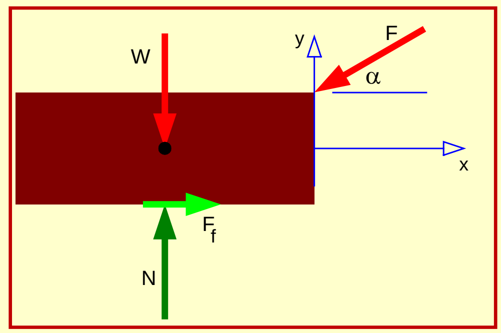


Fig. E10.4a

We recognize this problem is identical to Example 10.3, except the worker is pushing with a force that has components in the negative x and y directions. We account for this fact by deriving Eqs. (10.8) and (10.9) again. Following the same procedure, we obtain:

$$N = \frac{W}{1 - \mu \tan\alpha} \qquad (10.8a)$$

$$N = \frac{800}{1 - (0.44)\tan(28^\circ)} = 1{,}044 \text{ N} \qquad (a)$$

$$F = \frac{\mu W}{\cos\alpha(1 - \mu \tan\alpha)} \qquad (10.9a)$$

$$F = \frac{(0.44)(800)}{\cos(28^\circ)[1 - (0.44)\tan(28^\circ)]} = 520.4 \text{ N} \qquad (b)$$

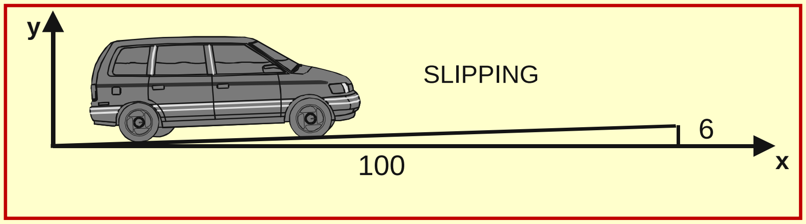
The result of 520.4 N is much greater than a force of 352.0 N, which would have been required if the worker had applied the force F in the direction of motion (with α = 0°). There are two reasons for the significant increase in the required force. First, the worker is pushing with a component of force downward on the crate, which increases the normal and friction forces. Second, only the horizontal component of the applied force is useful in sliding the crate.

EXAMPLE 10.5

Suppose you are driving in a storm that is depositing a mixture of snow and ice on the highways. As a prudent driver, you reduce your speed significantly, and are making slow but steady progress toward your destination. However, you encounter a long hill with a grade of 6%, as shown in Fig. E10.5. Unfortunately, part way up the hill your front-wheel-drive loses traction and your automobile stops. Your attempts to move the car forward fail because the tires spin as you depress the accelerator to increase the power applied to the wheels. In fact, your efforts cause the automobile to slip a short distance down the hill.

Determine the coefficient of friction that exists between the surface of the tire and the roadbed. Also explain why your attempts to move the automobile forward (up the hill) are futile.

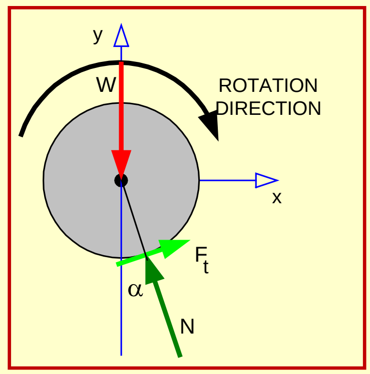
Fig. E10.5



Solution:

Let's begin our analysis by drawing a FBD of one of the front wheels as shown in Fig. E10.5a.

Fig. E10.5a



We select a front wheel for the FBD, because the automobile applies the traction force F_t to the pavement through the front wheels. Begin the analysis by writing the equilibrium relations.

$$\Sigma F_x = F_t \cos\alpha - N \sin\alpha = 0 \qquad \text{(a)}$$

$$\Sigma F_y = F_t \sin\alpha + N \cos\alpha - W = 0 \qquad \text{(b)}$$

Because the car slid down the hill, a condition of impending motion existed prior to the slide; hence, we write:

$$F_t = F_f = \mu N \qquad \text{(c)}$$

We have equated the traction force F_t with the friction force F_f.

Next, let's substitute Eq. (c) into Eq. (a) to give:

$$\mu N \cos\alpha - N \sin\alpha = 0 \qquad \text{(d)}$$

$$\mu = \tan\alpha = 0.06 \qquad \text{(e)}$$

The result in Eq. (e) shows that the friction coefficient is independent of the normal force N and is equal to the grade of the hill. This fact is evident, because N cancels out of Eq. (d). We will return to this point in the discussion following this solution.

To continue the analysis, substitute Eq. (c) into Eq. (b) and write:

$$\mu N \sin \alpha + N \cos \alpha - W = 0 \tag{f}$$

$$N = \frac{W}{\mu \sin \alpha + \cos \alpha} \tag{g}$$

We have determined the relation for N to present a complete solution. The coefficient of friction is independent of both N and the weight W acting on the wheel.

Attempts to begin moving an automobile on a snow and ice covered hill are often unsuccessful, because the wheels begin to spin as power is applied to the drive train. As the wheels spin, heat is generated, melting the snow and ice mixture. The water produced lubricates the surface of the ice and snow coating the highway, which reduces still further the coefficient of friction.

If you have ever driven much in snow and ice and have mastered the art of successfully negotiating slippery hills when all about you get stuck, the solution in Example 10.5 should certainly raise two very important questions.

1. Why do front wheel drive cars perform better in the snow and ice than rear drive cars?
2. Why do we spend significant amounts of money and put up with excessive road noise to equip our cars with snow tires?

Front wheel drive cars perform better in the snow than rear wheel drive cars, because more weight is placed over the drive wheels. The analysis showed that the friction coefficient was independent of the weight on the wheel, because we did not consider the possibility of the tire tread being forced into the snow and ice mixture covering the pavement. Higher wheel forces may in many cases produce enough penetration of the tread into the coating of snow to give some degree of interlocking at the tire-snow interface. The analysis in Example 10.5 considered the surfaces of the wheel and the pavement to be perfectly rigid, which is not valid.

Snow tires are also beneficial, because the tread is designed to penetrate snow and gain traction by interlocking with a coating of even relatively hard packed snow. However, the benefit gained by a tread that penetrates a coating of snow largely vanishes, if the pavement is coated with a rigid layer of ice.

10.4 FRICTION AND STABILITY

In many engineering applications friction is a culprit. It results in a loss of power, a loss of efficiency and the generation of undesirable heat. However, in other applications friction is a benefit that enables controlled movement and stability. The next series of examples demonstrate the role of friction in stability, preventing slipping, inducing tipping and promoting rolling.

10.4.1 Stability

When you climb a ladder, you want the ladder to remain stable or otherwise it slips and you fall. Stability is maintained, because friction prevents the ladder from slipping and sliding as you climb it. Most extension ladders are designed with rotating footpads with ridged rubber surfaces that improve frictional contact area, increase the coefficient of friction and enable these ladders to remain stable at smaller angles of inclination.

EXAMPLE 10.6

Consider a ladder resting on a level concrete surface, which leans against a wet, vinyl-clad wall, as shown in Fig. E10.6. A man with a mass of 88.0 kg climbs ¾ of the way up the ladder. We question the stability of the ladder. Is the ladder stable or does it slip and fall?

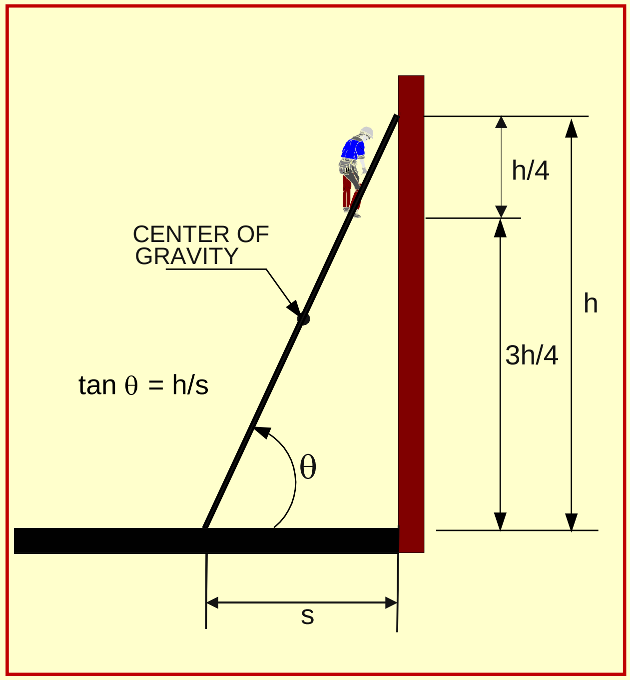


Fig. E10.6 Illustration of climbing a ladder.

Solution:

Before considering the ladder's stability, we need to gather more information about the ladder, its interactions with the floor and the wall and its weight. The FBD of the ladder, shown in Fig. E10.6a, provides some data before we seek the additional information necessary to determine if the ladder is stable.

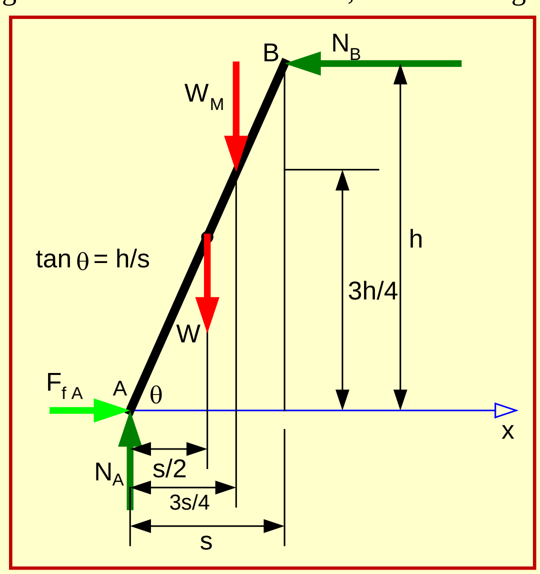


Fig. E10.6a

The parameters that control the stability of the ladder are listed below:

$$s = 3.5 \text{ m}; \quad h = 8 \text{ m}; \quad \theta = \tan^{-1}(h/s) = \tan^{-1}(8/3.5) = 66.37°; \quad W = 400 \text{ N};$$

$$W_M = (88)(9.807) = 863.0 \text{ N}; \quad \mu_A = 0.55; \quad \text{and } \mu_B = 0.$$

Note that we have assumed that $\mu_B = 0$, because a wet, vinyl-clad surface exhibits very little frictional restraint.

In performing a stability analysis, we assume that the ladder is about to slip (implying impending motion). This assumption is important because it permits us to determine the maximum friction force that can be developed to resist motion (instability).

Let's apply the three equilibrium relations that must be satisfied if the ladder is stable.

$$\Sigma F_y = N_A - W - W_M = 0 \qquad \text{(a)}$$

$$N_A = W + W_M = 400 + 863.0 = 1263.0 \text{ N} \qquad \text{(b)}$$

$$\Sigma M_A = N_B\,(h) - W\,(s/2) - W_M\,(3s/4) = 0 \qquad \text{(c)}$$

$$N_B = (s)(2W + 3W_M)/(4h) = (3.5)[2(400) + 3(863.0)]/[4(8)] = 370.7 \text{ N} \qquad \text{(d)}$$

$$\Sigma F_x = F_{fA} - N_B = 0 \qquad \text{(e)}$$

$$F_{fA} = N_B = 370.7 \text{ N} \qquad \text{(f)}$$

This result indicates that the friction force F_{fA}, at the base of the ladder, must be 370.7 N to maintain the ladder in a state of stable equilibrium. The maximum friction force that can develop at Point A is given by Eq. (10.3) as:

$$(F_{fA})_{Max} = \mu_A\, N_A = (0.55)(1{,}263) = 694.7 \text{ N} \qquad \text{(g)}$$

Because $F_{fA} = 370.7$ N $\leq (F_{fA})_{Max} = 694.7$ N,, the ladder will be stable under the conditions described above. The magnitude of the friction force that can develop at the base of the ladder is larger than the force necessary to maintain equilibrium.

EXAMPLE 10.7

Let's consider the ladder shown in Fig. E10.7, and determine its stability if all the conditions are the same as described in Example 10.6 except for the following parameters:

$$h = 7.5 \text{ m}; \; s = 5.2 \text{ m}; \; \mu_A = 0.38, \text{ and } \mu_B = 0$$

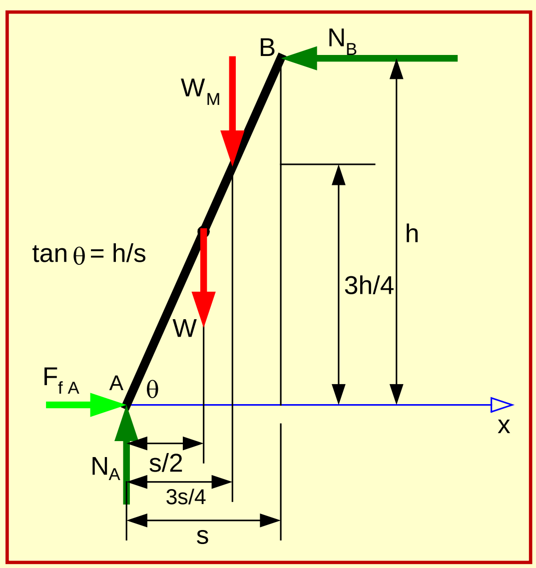


Fig. E10.7

Solution:

Utilizing the FBD presented in Fig. E10.7, we explore the stability question by writing the equilibrium relations:

$$\Sigma F_y = N_A - W - W_M = 0 \qquad \text{(a)}$$

$$N_A = W + W_M = 400 + 863.0 = 1{,}263 \text{ N} \qquad \text{(b)}$$

$$\Sigma M_A = N_B\,(h) - W\,(s/2) - W_M\,(3s/4) \qquad \text{(c)}$$

$$N_B = (s)(2W + 3W_M)/(4h) = (5.2)[2(400) + 3(863.0)]/[4(7.5)] = 587.4 \text{ N} \qquad \text{(d)}$$

$$\Sigma F_x = F_{fA} - N_B = 0 \qquad \text{(e)}$$

Substituting the results from Eq. (d) into Eq. (e) gives:

$$F_{fA} = N_B = 587.4 \text{ N} \qquad \text{(f)}$$

Next, we must determine if the friction force F_{fA} = 587.4 N at the base of the ladder can be developed. From Eq. (10.3), $(F_{fA})_{Max}$ is given by:

$$(F_{fA})_{Max} = \mu_A\, N_A = (0.38)(1{,}263) = 479.9 \text{ N} \qquad \text{(g)}$$

Clearly, the ladder's feet will slide away from the wall and the ladder will fall because:

$$F_{fA} = 587.4 \text{ N} > (F_{fA})_{max} = 479.9 \text{ N} \qquad \text{(h)}$$

EXAMPLE 10.8

Let's consider a slightly more complex question pertaining to the stability of the ladder shown in Fig. E10.6. The new parameters describing the condition of the ladder are:

$$s = 3.5 \text{ m}; \quad h = 8 \text{ m}; \quad \theta = \tan^{-1}(h/s) = \tan^{-1}(8/3.5) = 66.37°; \quad W = 380 \text{ N};$$

$$W_M = (88)(9.807) = 863.0 \text{ N}; \quad \mu_A = 0.45; \quad \text{and } \mu_B = 0.22$$

Solution:

Because the coefficient of friction at the point of contact of the ladder with the wall is not zero, you must modify the FBD by adding a friction force F_{fB}, as shown in Fig. E10.8. Examination of the FBD indicates that four unknown forces develop, namely N_A, N_B, F_{fA}, and F_{fB}. However, we only have three equations of equilibrium to apply in our stability analysis.

To resolve this difficulty, let's assume that the ladder is in a condition of impending motion at the wall (point B), which enables us to use Eq. (10.3) and express F_{fB} in terms of N_B. This assumption reduces the number of unknown forces to three, which is equal to the number of relevant equilibrium relations.

$$F_{fB} = (F_{fB})_{Max} = \mu_B\, N_B \qquad \text{(a)}$$

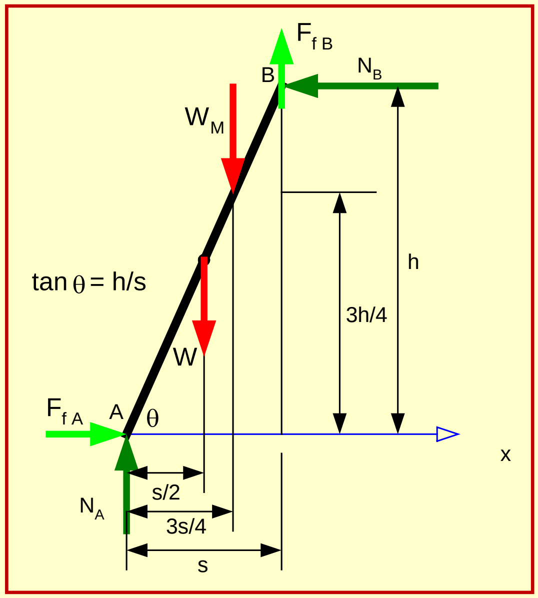


Fig. E10.8

Applying the equilibrium relations yields:

$$\Sigma M_A = -\,W\,(s/2) - W_M\,(3s/4) + F_{fB}\,(s) + N_B\,(h) = 0 \qquad \text{(b)}$$

Substituting Eq. (a) into Eq. (b), and solving for N_B gives:

$$N_B = (s/4)(2W + 3W_M\,)/(\mu_B\,s + h) \qquad \text{(c)}$$

$$N_B = (3.5/4)[2(380) + 3(863.0)]/[(0.22)(3.5) + 8] = 334.1\text{ N} \qquad \text{(d)}$$

$$\Sigma F_y = N_A - W - W_M + F_{fB} = 0 \qquad \text{(e)}$$

Using Eq. (a) and solving Eq. (e) for N_A yields:

$$N_A = W + W_M - \mu_B\,N_B = 380 + 863.0 - (0.22)(334.1) = 1{,}169.5\text{ N} \qquad \text{(f)}$$

Next, we employ the final equilibrium relation to determine the friction force at the base of the ladder required to insure stability.

$$\Sigma F_x = F_{fA} - N_B = 0 \qquad \text{(g)}$$

$$F_{fA} = N_B = 334.1\text{ N} \qquad \text{(h)}$$

From Eq. (10.3) and the condition of impending motion at point A, we write:

$$(F_{fA})_{Max} = \mu_A\,N_A = (0.45)(1{,}169.5) = 526.3\text{ N} \qquad \text{(i)}$$

Then we compare the maximum friction force possible with the required friction force as indicated below:

$$F_{fA} = 334.1\text{ N} \leq (F_{fA})_{Max} = 526.3\text{ N} \qquad \text{(j)}$$

We conclude that the ladder is stable, and with a significant margin of safety.

Let's use another approach to determine if the ladder in Example 10.8 is stable or if it will be unstable and slip. If the ladder is about to slip, we have impending motion at both points A and B and can write:

$$F_{fA} = (F_{fA})_{Max} = \mu_A N_A \qquad \text{(a)}$$

$$F_{fB} = (F_{fB})_{Max} = \mu_B N_B \qquad \text{(b)}$$

With this assumption, we have two unknowns N_A and N_B and three equilibrium equations. Next, write the two equilibrium equations for the Cartesian forces as:

$$\sum F_x = F_{fA} - N_B = 0 \qquad \text{(c)}$$

Using Eq. (a) and solving:

$$F_{fA} = \mu_A N_A = N_B \qquad \text{(d)}$$

and for $\sum F_y$

$$\sum F_y = N_A - W - W_M + F_{fB} = 0 \qquad \text{(e)}$$

Simplifying Eq. (e) and substituting numerical values for W and W_M, we obtain:

$$N_A + \mu_B N_B = W + W_M = 380 + 863 = 1{,}243 \text{ N} \qquad \text{(f)}$$

Substituting Eq. (d) into Eq. (f) yields:

$$N_B [(1/\mu_A) + \mu_B] = 1{,}243 \text{ N} \qquad \text{(g)}$$

Using μ_A and μ_B and solving Eq. (g) for N_B yields:

$$N_B = 509.0 \text{ N} \qquad \text{(h)}$$

Substituting Eq. (h) into Eq. (f) and solving for N_A gives:

$$N_A = 1131.0 \text{ N} \qquad \text{(i)}$$

Finally writing the equation for moments about point A gives:

$$\sum M_A = F_{fB} (s) + N_B (h) - W_M (3s/4) - W (s/2) \qquad \text{(j)}$$

Using the results from Eqs. (h), (i) and (b) as well as s, h, W, and W_M, we obtain:

$$\sum M_A = 1{,}534 \text{ N-m} \qquad \text{(k)}$$

The positive value for $\sum M_A$ indicates that the ladder should have the tendency to rotate in the counterclockwise direction. This direction is not consistent with the ladder slipping at both points A and B and rotating clockwise as it falls. Hence, we conclude that the ladder is stable, does not slip at either contact point, and the assumptions made in writing Eqs. (a) and (b) are **not** valid.

10.4.2 <u>Tipping</u>

Let's consider a different stability problem that involves sliding objects along some surface. When a force is applied to an object to overcome the frictional forces, it may slide or it may tip. We seek to determine if it will slide or tip for a given set of conditions describing the object. Let's introduce this topic by considering a rectangular box resting on a horizontal surface, as illustrated in Fig. 10.10.

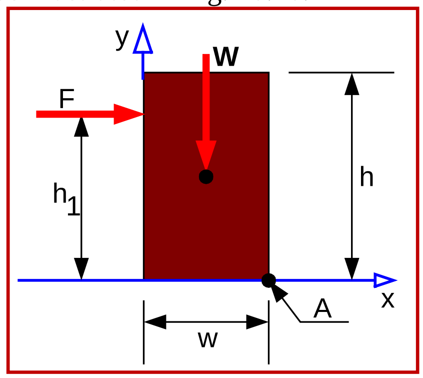


Fig. 10.10 A rectangular box sliding along a horizontal surface.

EXAMPLE 10.9

Consider a crate that weighs 600 lb resting on a level concrete floor. The coefficient of friction μ = 0.40 between the floor and the box. The dimensions of the quantities defined in Fig. 10.10 are h = 6 ft, h_1 = 4.5 ft, and w = 1.6 ft. Determine the force required to move the crate, and establish if the crate will slide or tip when the force F is applied.

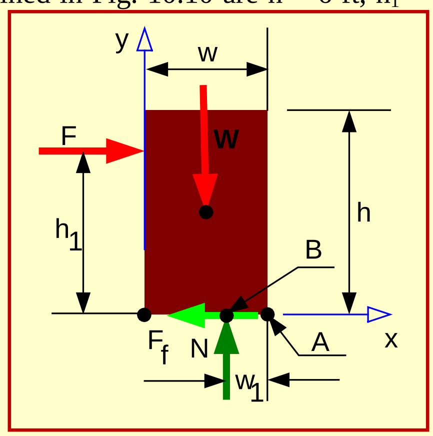


Fig. E10.9

Solution:

Begin with a FBD of the crate as shown in Fig. E10.9. Next, apply the equations of equilibrium.

$$\Sigma F_y = N - W = 0 \qquad (a)$$

$$N = W = 600 \text{ lb}$$

$$\Sigma F_x = F - F_f = 0 \qquad (b)$$

$$\Sigma M_B = W\,[(w/2) - w_1] - F\,(h_1) = 0 \qquad (c)$$

At this stage, we have solved for one unknown (N); however, three unknowns remain (F, F_f and w_1) with only two applicable equations remaining. We must consider each case of impending motion separately to determine if the crate will tip or slide.

First, assume that the crate slides with $F_{Slide} \leq (F_f)_{Max}$ Then the following relation holds:

$$F_{Slide} = F_f = \mu\ N = (0.40)(600) = 240 \text{ lb} \qquad \text{(d)}$$

Now, let's consider the argument for tipping. When the crate begins to tip, the point of application of N (point B) shifts to the right hand edge of the crate (point A). For this situation, $w_1 = 0$ and Eq. (c) becomes:

$$F_{Tip} = (W\ w)/(2\ h_1) = [(600)(1.6)]/[(2)(4.5)] = 106.7 \text{ lb} \qquad \text{(e)}$$

Clearly, the crate will tip before it slides because $F_{Tip} < F_{Slide}$.

EXAMPLE 10.10

The crate in Example 10.9 tipped, because we applied the force F near the top of the crate. For the same conditions, determine the largest value of h_1 for applying the force F, which will result in the crate sliding without tipping.

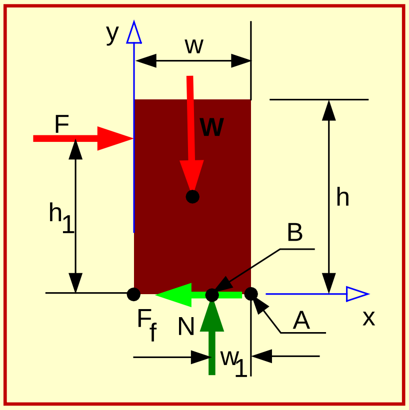


Fig. E10.10

Solution:

To solve this problem, recognize that tipping becomes possible when the point of application of the normal force N shifts to point A in Fig. E10.10, and $w_1 = 0$. Recall Eq. (c) from Example 10.9, let $w_1 = 0$, and then solve for h_1 to obtain:

$$h_1 = (W\ w)/(2\ F) \qquad \text{(a)}$$

$$h_1 = [(600)(1.6)]/[(2)(240)] = 2.00 \text{ ft} \qquad \text{(b)}$$

For $h_1 < 2.00$ ft, the crate will slide.

10.4.3 <u>Rolling</u>

Let's consider the role of friction in determining if a cylindrical object rolls or slides. To illustrate this situation, refer to the roller positioned adjacent to a step, as shown in Fig. 10.11.

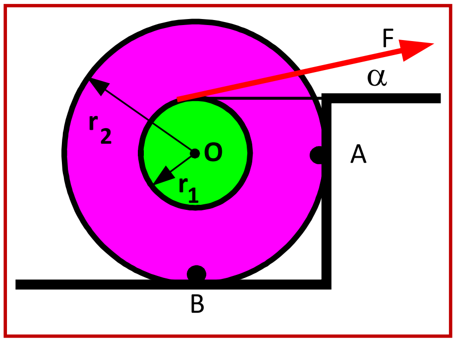


Fig. 10.11 Rolling a cylinder over a step.

As we apply a force F to a cable wound about the smaller cylinder, the larger roller contacts the step. If the friction force developed at point A is sufficiently large, the cylinders will roll up the vertical face and the assembly will negotiate the step. On the other hand, if the friction force is insufficient, the cylinder will slip as it rotates and it will remain at the bottom of the step.

Rolling friction can be both beneficial and detrimental in the same application. For example, when designing wheels for an automobile, it is necessary to maintain sufficient rolling friction to provide the thrust forces that are generated during acceleration. However, this rolling friction reduces the fuel efficiency of the engine by resisting the automobile's forward motion, when it is traveling at a constant velocity.

EXAMPLE 10.11

Consider a cylindrical arrangement similar to that shown in Fig. 10.11, with a force F applied at an angle $\alpha = 25°$. If the coefficient of friction at the contact points A and B is $\mu_A = \mu_B = 0.45$, determine if the cylinder will roll up and over the step or slip and remain in the corner as the force is increased. Also determine the force required to roll a cylindrical arrangement that weighs 900 N up and over the step. Note that $r_1 = 150$ mm, and $r_2 = 450$ mm.

Solution:

In constructing the FBD shown in Fig. E10.11, we have assumed that the cylindrical assembly has been lifted from the horizontal surface ($N_B = 0$) and is beginning to roll up the step. To resolve the roll or slip question, write the equilibrium relations to obtain:

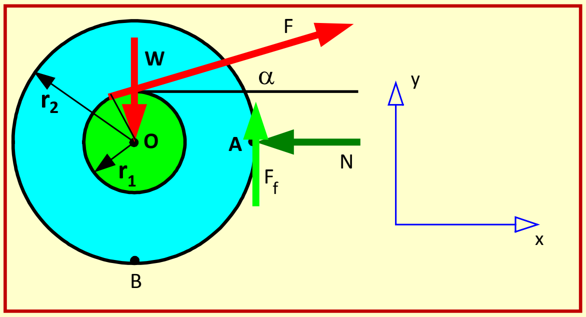


Fig. E10.11

$$\sum M_O = F_f\,(r_2) - F\,(r_1) = 0 \qquad \text{(a)}$$

$$F_f = (r_1/r_2)\,F = (150/450)F = 0.3333\,F \qquad \text{(b)}$$

$$\Sigma F_x = F\cos\alpha - N = 0; \qquad \Rightarrow \qquad N = 0.9063\,F \qquad \text{(c)}$$

The maximum possible value of the friction force at point A is given by Eq. (10.3) as:

$$(F_f)_{Max} = \mu\,N = (0.45)(0.9063)F = 0.4078\,F \qquad \text{(d)}$$

A comparison of the results of Eq. (b) and Eq. (d) shows that:

$$F_f = 0.3333\,F < (F_f)_{Max} = 0.4078\,F \qquad \text{(e)}$$

Clearly, the cylinder will roll up and over the step, because $F_f < (F_f)_{Max}$.

Finally, let's determine the force F required.

$$\Sigma F_y = F \sin\alpha - W + F_f = 0 \qquad (f)$$

Substituting the results of Eq. (b) into Eq. (f) yields:

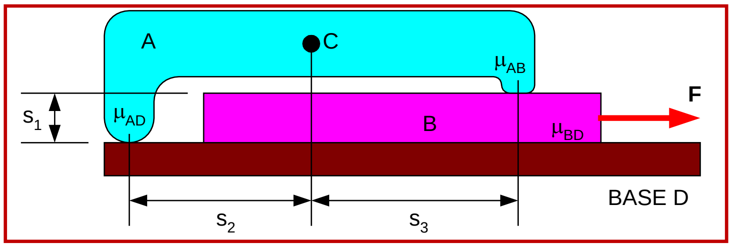
$$0.4226\ F + (0.3333\ F) = 900 \quad \Rightarrow \quad F = 1{,}191\ N \qquad (g)$$

Because $(F_f) < (F_f)_{Max}$, the cylindrical assembly will roll up the step without slipping.

10.4.4 Multiple Body Interactions

In some cases two or more bodies are connected either with links, springs or by contact. A force is applied to one of the bodies in the assembly, and the problem is to determine which body will move and the magnitude of the force required to initiate this motion. An example of a two body problem is presented in Fig. 10.12. With two bodies there are two options for motion that depend on the distribution of the contact forces and the coefficients of friction at the contact points or contact areas. Solutions require that both options be explored and the scenario that yields the minimum force to overcome friction is selected as the correct solution.

Fig. 10.12 A two body friction example with three coefficients of friction.



EXAMPLE 10.12

Consider the two bodies, shown in Fig. 10.12, and determine the maximum force F that can be applied to block B before it moves. Let the controlling parameters be defined as:

W_A = 60 lb, W_B = 36 lb, s_1 = 12 in, s_2 = 16 in. s_3 = 28 in., μ_{AB} = 0.5, μ_{AD} = 0.3, and μ_{BD} = 0.6.

Solution:

Recognize that there are two possible outcomes when a force F is applied to block B. Block A and B can slide together on the base, or block B can slide from under block A along the surface of the base D. We solve for both options and select the solution yielding the minimum value of the force F that initiates motion. For **option 1**, assume blocks A and B slide together on the base.

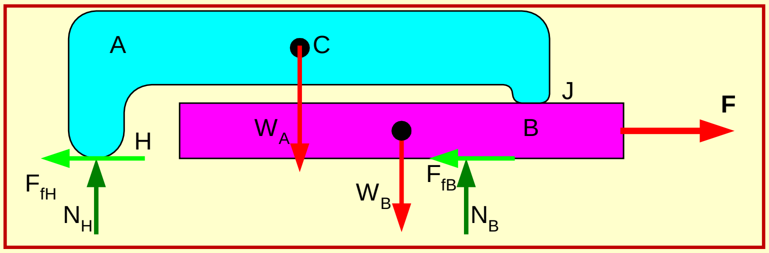
Let's begin by drawing a FBD of block A and B as an assembly, as shown in Fig. E10.12a

Next write the equilibrium relation $\sum F_x = 0$ assuming impending motion of the assembly:

$$\sum F_x = F - F_{fH} - F_{fB} = 0 \qquad (a)$$

Next let's draw the FBD of member A, as shown in Fig. E10.12b, and consider the two possible options.

Fig. E10.12a



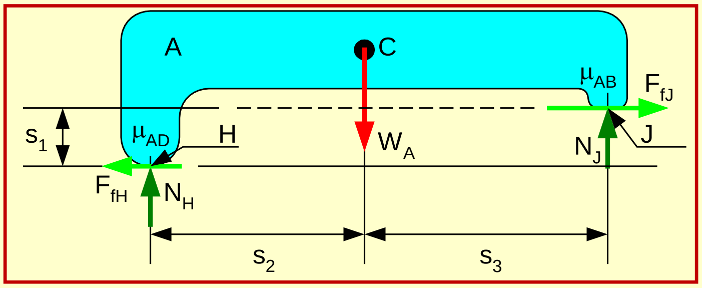


Fig. E10.12b

Assume block A and B slide together along the base D and that motion is impending at point H. Write the equilibrium relations and the friction equation for Block A to obtain:

$$\sum F_y = N_H + N_J - W_A = 0 \tag{b}$$

$$\sum M_J = -N_H (s_2 + s_3) - F_{fH} (s_1) + W_A (s_3) = 0 \tag{c}$$

$$F_{fH} = \mu_{AD} N_H = 0.3 N_H \tag{d}$$

Substituting Eq. (d) and numerical values into Eq. (c) yields:

$$N_H (16 + 28) + (12)(0.3) N_H - (60)(28) = 0 \tag{e}$$

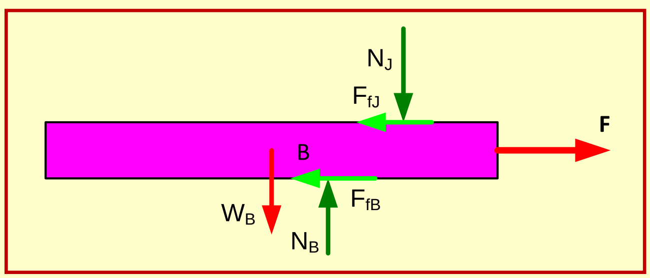
$$N_H = 35.29 \text{ lb} \tag{f}$$

Combining Eqs. (f) and (b) gives:

$$N_J = W_A - N_H = 60 - 35.29 = 24.71 \text{ lb} \tag{g}$$

Draw the FBD of block B as shown in Fig. E10.12c:

Fig. E10.12c



Write the equilibrium relation:

$$\sum F_y = N_B - N_J - W_B = 0 \tag{h}$$

$$N_B = 36 + 24.71 = 60.71 \text{ lb} \tag{i}$$

Finally recall Eq. (a) and Eqs. (f) and (i) to obtain:

$$F = \mu_{AD}\, N_H + \mu_{BD}\, N_B = (0.3)(35.29) + (0.6)(60.71) = 47.01 \text{ lb} \qquad \text{(j)}$$

Option 2:

Assume block B slides along the base D while block A remains fixed to the base, and that motion is impending at points J and B but not at point H. Consider the FBD of block A shown again in Fig. E10.12b. Write the equilibrium relations and the friction equation to obtain:

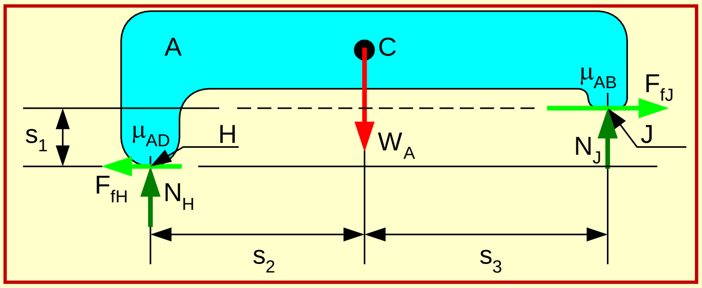


Fig. E10.12b

$$\sum F_y = N_H + N_J - W_A = 0 \qquad \text{(k)}$$

$$\sum M_H = N_J\,(s_2 + s_3) - F_{fJ}\,(s_1) - W_A\,(s_2) = 0 \qquad \text{(l)}$$

$$F_{fJ} = \mu_{AB}\, N_J = 0.5\, N_J \qquad \text{(m)}$$

Substitute Eq. (m) into Eq. (l) to obtain:

$$(16 + 28)N_J - (0.5)(12)N_J - (16)(60) = 0 \qquad \text{(n)}$$

$$N_J = 25.26 \text{ lb} \qquad \text{(o)}$$

Substituting Eq. (o) into Eq. (k) yields:

$$N_H = 60 - 25.26 = 34.74 \text{ lb} \qquad \text{(p)}$$

Reconsider the FBD in Fig. E10.12c and write the equilibrium and friction equations assuming impending motion of block B.

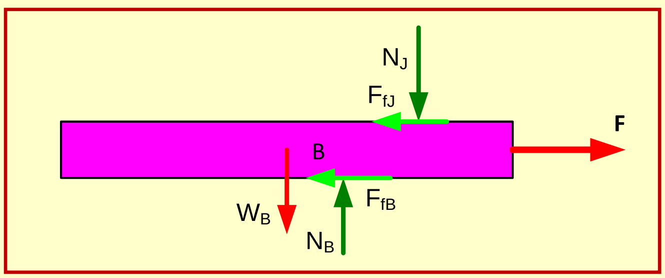


Fig. E10.12c

$$\sum F_y = N_B - W_B - N_J = 0 \qquad \text{(q)}$$

$$N_B = W_B + N_J = 36 + 25.26 = 61.26 \text{ lb} \qquad \text{(r)}$$

$$\sum F_x = F - F_{fJ} - F_{fB} = 0 \qquad \text{(s)}$$

Note, assuming impending motion of block B gives:

$$F_{fJ} = \mu_{AB}\, N_J = 0.5\, N_J \qquad \text{and} \qquad F_{fB} = \mu_{BD}\, N_B = 0.6\, N_B \qquad (t)$$

Finally substituting Eq. (t) into Eq. (s) yields:

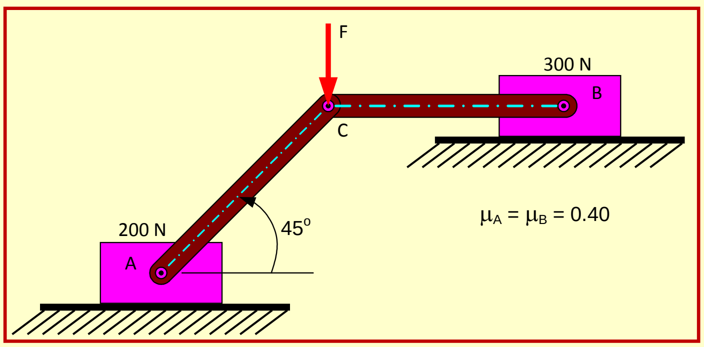
$$F = F_{fJ} + F_{fB} = (0.5)(25.26) + (0.6)(61.26) = 49.39 \text{ lb} \qquad (u)$$

Option 1 with F = 47.01 lb is less that the result of 49.39 lb for option 2; hence, we conclude that that blocks A and B slide together over the base D.

EXAMPLE 10.13

Blocks A and B, located at different levels, are connected by two links AC and BC, as shown in Fig. E10.13. A vertical force F is applied at the pin connecting the two links. Determine the maximum magnitude of the force F before one of the blocks begins to move. The weight of block A is 200 N and the weight of block B is 300 N. The coefficient of friction $\mu_A = \mu_B = 0.40$

Fig. E10.13

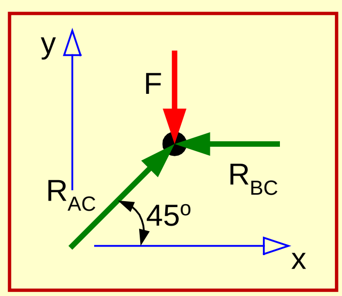


Solution:

As the force F is increased, the links exert horizontal forces on blocks A and B until the maximum possible friction force on one or the other of the two blocks is exceeded. We will consider the forces acting on each block to determine which of the two blocks moves with the minimum force F.

Draw a FBD of pin C as shown in Fig. E10.13a, and write the equilibrium equations for the pin.

Fig. E10.13a



$$\sum F_x = R_{AC} \cos (45^o) - R_{BC} = 0 \qquad (a)$$

$$\sum F_y = R_{AC} \sin (45^o) - F = 0 \qquad (b)$$

Solving Eq. (a) and Eq. (b) gives:

$$R_{BC} = F \qquad \text{and} \qquad R_{AC} = 1.414\, F \qquad (c)$$

Option 1: Determine force F to overcome the maximum friction force at block B. Draw the FBD of block B, as shown in Fig. E10.13b.

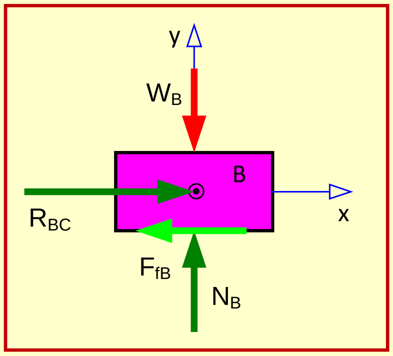


Fig. E10.13b

Writing the equilibrium relations gives:

$$\sum F_x = R_{BC} - F_{fB} = 0 \qquad \text{(d)}$$

$$\sum F_y = N_B - W_B = 0 \qquad \text{(e)}$$

Solving Eqs. (d) and (e) yields:

$$N_B = W_B = 300 \text{ N} \qquad \text{and} \qquad R_{BC} = F_{fB} = F \qquad \text{(f)}$$

If block B develops the maximum friction force at impending motion, we write:

$$F_{fB} = \mu_B N_B = (0.40)(300) = 120.0 \text{ N} \qquad \text{(g)}$$

Substituting Eq. (g) into Eq. (f) gives:

$$F = 120.0 \text{ N} \qquad \text{(h)}$$

Option 2: Determine force F to overcome the maximum friction force at block A. Draw the FBD of block A, as shown in Fig. E10.13c.

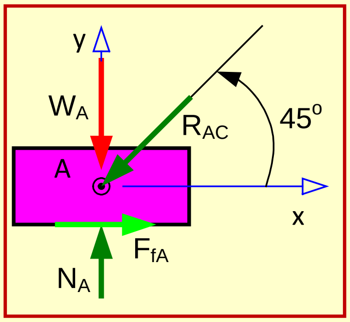


Fig. E10.13c

Writing the equilibrium relations gives:

$$\sum F_x = F_{fA} - R_{AC} \cos(45^o) = 0 \qquad \text{(i)}$$

$$\sum F_y = N_A - W_A - R_{AC} \sin(45^o) = 0 \qquad \text{(j)}$$

Solving Eqs. (i) and (j) and recalling R_{AC} = 1.414 F from Eq. (c) yields:

$$F_{fA} = (1.414 \text{ F}) \cos(45^o) = F \qquad \text{(k)}$$

$$N_A = W_A + (1.414 \text{ F}) \sin(45^o) = 200 + F \qquad \text{(l)}$$

If block A develops the maximum friction force at impending motion, we write:

$$F_{fA} = \mu_A N_A = (0.40)(200 + F) = 80 + 0.40\,F \qquad (m)$$

Substituting Eq. (k) into Eq. (m), we solve for F to obtain:

$$F = 133.3\text{ N} \qquad (n)$$

Comparing the results from options 1 and 2 indicates that block B moves when the force on pin C is 120.0 N and block A remains at rest.

EXAMPLE 10.14

A wedge mechanism is illustrated in Fig. E10.14. It is comprised of a clamp with a tapered jaw and a rigid length of tube. A flat piece of sheet metal with a weight W is clamped between the tube and the vertical side of the clamp. If θ is the angle of the taper and the tube's weight is so small it can be neglected, derive an equation for the minimum value of the friction coefficient for the clamp to hold the sheet metal regardless of its weight. Assume the coefficient of friction is identical at all contact lines or areas.

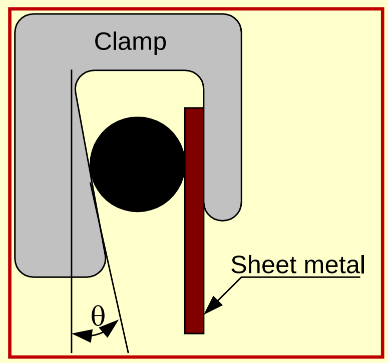


Fig. E10.14

Solution:

Draw a FBD of the sheet metal held in the clamp by friction forces, as shown in Fig. E10.14a, and write the equilibrium equations and the friction equation assuming impending motion:

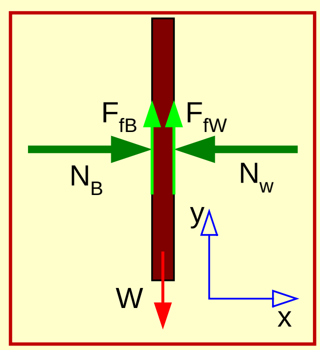


Fig. E10.14a

Writing the equilibrium relations for the sheet metal gives:

$$\sum F_x = N_B - N_w = 0 \quad \rightarrow \quad N_B = N_w = N \qquad (a)$$

$$\sum F_y = F_{fB} + F_{fw} - W = 0 \qquad (b)$$

By assuming impending motion of the sheet metal, the friction forces can be written as:

$$F_{fB} = \mu\, N_B = \mu\, N$$
$$F_{fw} = \mu\, N_w = \mu\, N \qquad \text{(c)}$$

Substituting Eq. (c) into Eq. (b) yields:

$$F_{fB} + F_{fw} = W = 2(\mu\, N) \quad \rightarrow \quad N = W/(2\mu) \qquad \text{(d)}$$

Next draw a FBD of the tube, as shown in Fig. E10.14b.

Writing the equilibrium relations for the tube gives:

$$\sum F_x = N_C \cos\theta + F_{fC} \sin\theta - N_B = 0 \qquad \text{(e)}$$

$$\sum F_y = N_C \sin\theta - F_{fC} \cos\theta - F_{fB} = 0 \qquad \text{(f)}$$

$$\sum M_O = (r)\, F_{fC} - (r)\, F_{fB} = 0 \qquad \text{(g)}$$

where r is the outer radius of the tube.

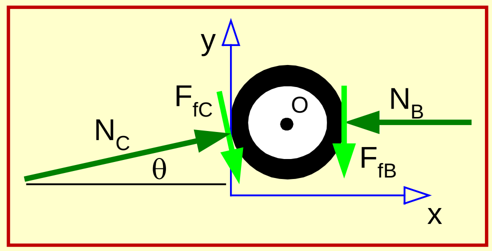


Fig. E10.14b

It is evident from Eq. (g) that:

$$F_{fC} = F_{fB} \qquad \text{(h)}$$

Assuming impending motion for the friction force F_{fB} yields:

$$F_{fB} = \mu\, N_B = \mu\, N \qquad \text{(i)}$$

It is clear from Eq. (h) that the two friction forces are equal. Combining Eqs. (h) and (i), we can write:

$$F_{fC} = F_{fB} = \mu\, N \qquad \text{(j)}$$

Substituting Eqs. (j) and (a) into Eqs. (e) and (f) yields:

$$N_C \cos\theta + (\mu\, N) \sin\theta = (N) \qquad \text{(k)}$$

$$N_C \sin\theta - (\mu\, N) \cos\theta = (\mu\, N) \qquad \text{(l)}$$

Multiply Eq. (k) by sin θ and Eq. (l) by cos θ and subtract to eliminate N_C and obtain:

$$\mu\, N (\sin^2\theta + \cos^2\theta) = N (\sin\theta - \mu \cos\theta) \qquad \text{(m)}$$

Recall the trigonometric identity:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{(n)}$$

Using Eq. (n) in Eq. (m) and further simplifying gives the result:

$$\mu = \frac{\sin\theta}{(1+\cos\theta)} \qquad \text{(o)}$$

EXAMPLE 10.15

A wedge mechanism is illustrated in Fig. E10.15. It is comprised of a clamp with a tapered jaw and a rigid cylinder. A flat piece of sheet metal with a weight W is clamped between the cylinder and the vertical side of the clamp. If the angle of the taper θ is equal to 45° and the cylinder's weight W_C is 48 lb, determine the maximum weight of the sheet metal that the clamp can support. Assume the coefficient of friction at points A and B is $\mu_A = \mu_B = 0.35$. The coefficient of friction at point C is unknown.

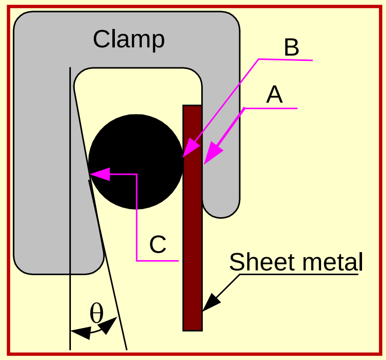


Fig. E10.15

Solution:

Draw a FBD of the sheet metal, as shown in Fig. E10.15a, and write $\sum F_x = 0$, and $\sum F_y = 0$.

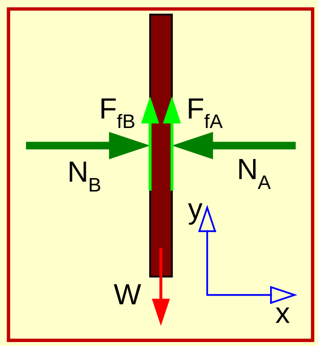


Fig. E10.15a

$$\sum F_x = N_B - N_A = 0 \qquad \text{(a)}$$

Then:

$$N_A = N_B = N \qquad \text{(b)}$$

$$\sum F_y = F_{fA} + F_{fB} - W = 0 \qquad \text{(c)}$$

When $W = W_{Max}$, which is the heaviest metal sheet the clamp can support, motion is impending at both points A and B; hence, we write the friction relation as:

$$F_{fA} = F_{fB} = \mu_B N \qquad \text{(d)}$$

and from Eqs. (c) and (d):

$$W = 2(\mu_B N) \qquad \text{(e)}$$

Consider the FBD of the cylinder, as shown in Fig. E10.15b.

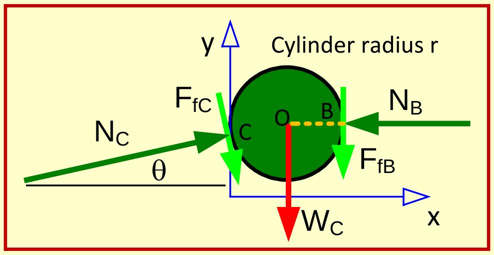


Fig. E10.15b The cylinder has a tendency to rotate in the clockwise direction about point C.

Writing the three valid equations of equilibrium yields:

$$\sum M_C = r[-W_C \cos\theta - F_{fB}(1 + \cos\theta) + N_B \sin\theta] = 0 \qquad \text{(f)}$$

or

$$N_B \sin\theta - F_{fB}(1 + \cos\theta) = W_C \cos\theta \qquad \text{(g)}$$

Substitute Eq. (d) into Eq. (g) and solve the resulting expression for N_B to give:

$$N_B = \frac{W_C \cos\theta}{\sin\theta - \mu_B(1+\cos\theta)} \qquad \text{(h)}$$

Next write the equilibrium equations $\sum F_x = 0$ and $\sum F_y = 0$ using the FBD in Fig. E10.15b.

$$\sum F_x = N_C \cos\theta + F_{fC} \sin\theta - N_B = 0 \qquad \text{(i)}$$

and

$$\sum F_y = N_C \sin\theta - F_{fC} \cos\theta - F_{fB} - W_C = 0 \qquad \text{(j)}$$

Using Eq. (i) to solve for F_{fC} gives:

$$F_{fC} = (N_B - N_C \cos\theta)/\sin\theta \qquad \text{(k)}$$

Substitute Eq. (k) into Eq. (j) and solving for N_C yields:

$$N_C = W_C \sin\theta + N_B(\cos\theta + \mu_B \sin\theta) \qquad \text{(l)}$$

Next substitute Eq. (h) into Eq. (l) and derive the relation for N_C as:

$$N_C = \frac{W_C(1-\mu_B \sin\theta)}{\sin\theta - \mu_B(1+\cos\theta)} \qquad \text{(m)}$$

Note meaningful solutions are obtained if and only if the quantity $[\sin\theta - \mu_B(1 + \cos\theta)]$ is greater than zero, because N_C cannot become negative.

Let $\theta = 45°$, $W_C = 48$ lb and $\mu_B = 0.35$ and substitute these quantities into Eq. (h) and Eq. (m) to give:

$$N_B = N = 309.6 \text{ lb}$$
$$N_C = 329.5 \text{ lb} \qquad \text{(m)}$$

From Eq. (e) we write:

$$W_{Max} = 2(\mu_B\, N) = 2(0.35)(309.6) = 216.7 \text{ lb} \qquad \text{(n)}$$

To examine the friction coefficient at point C, let's consider the FBD in Fig. 10.15b and write $\sum M_O = 0$, where O is the center of the cylinder. From $\sum M_O = 0$, it is evident that:

$$F_{fC} = F_{fB} \qquad \text{(o)}$$

From Eqs. (d) and (o), we write:

$$F_{fC} = F_{fB} = \mu_B\, N = 0.35(309.6) = 108.4 \text{ lb} \qquad \text{(p)}$$

Assume impending motion at point C and solve for the friction coefficient μ_C as:

$$\mu_C = F_{fC}/N_C = 108.4/329.5 = 0.3290 \qquad \text{(q)}$$

If the coefficient of friction at point C is less than 0.3290 the cylinder will slide instead of rolling.

10.5 FRICTION EFFECTS ON WEDGES

A wedge is classified as a simple machine, because it can be used to amplify a force if the effects of friction are not too large. Lifting a block with a weight W by using a tapered wedge is illustrated in Fig. 10.13. For the wedge to act as a machine, the force F used to drive the wedge under the block must be less than the weight W.

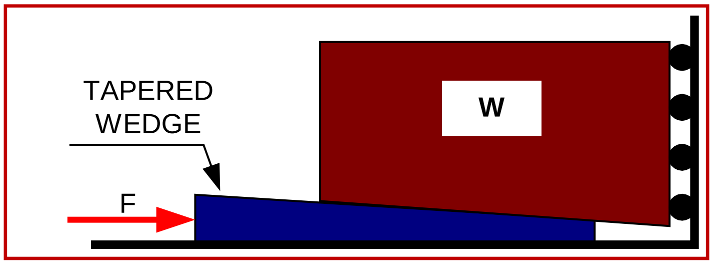


Fig. 10.13 A tapered wedge acts as a machine to lift a weight W.

The use of a wedge as a simple machine has three modes of operation:

- Lifting
- Holding
- Lowering

Friction plays an important role in all three of these operations.

10.5.1 Lifting with a Wedge

Let's first consider the lifting operation and explore the role of friction. As usual, we begin with the FBD diagram of the block and the wedge shown in Fig. 10.14. Note that the force F in the figure acts in the positive x direction on the wedge, as the block is lifted. Also, we assume that the rollers eliminate the friction force on the vertical surface of the block.

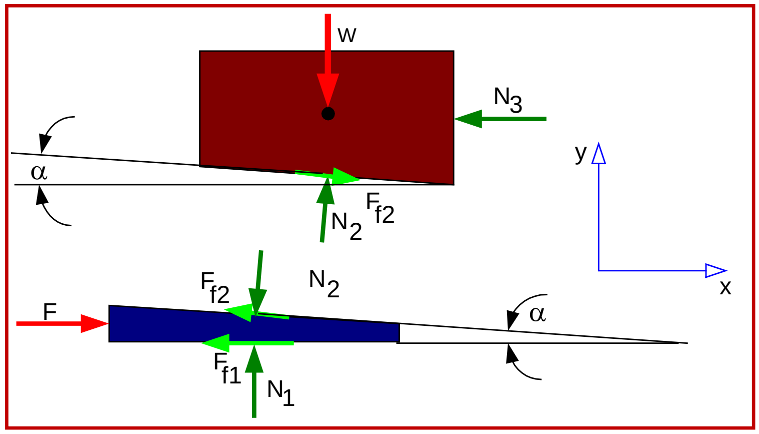


Fig. 10.14 FBD of a block and wedge in a lifting operation.

Let's apply the equations of equilibrium to the block and then to the wedge to determine the relation between the applied force F and the weight W. For the block, we write:

$$\Sigma F_y = N_2 \cos\alpha - F_{f2} \sin\alpha - W = 0 \tag{a}$$

Because motion occurs as the wedge moves, we may write:

$$F_{f1} = \mu N_1 \quad \text{and} \quad F_{f2} = \mu N_2 \tag{b}$$

Combining Eqs. (a) and (b) and solving for N_2, the normal force acting on the block, yields:

$$N_2 = \frac{W}{\cos\alpha - \mu \sin\alpha} \tag{c}$$

For the wedge one of the equilibrium equations gives:

$$\Sigma F_y = N_1 + F_{f2} \sin\alpha - N_2 \cos\alpha = 0 \tag{d}$$

Combining Eqs. (b), (c) and (d) yields:

$$N_1 = W \tag{e}$$

Applying the final relevant equilibrium relation $\Sigma F_x = 0$ for the wedge gives:

$$\Sigma F_x = F - F_{f1} - F_{f2} \cos\alpha - N_2 \sin\alpha = 0 \tag{f}$$

Substituting Eqs. (b), (c), and (e) into Eq. (f) and solving for the force F yields:

$$F = W\left[\frac{2\mu + (1 - \mu^2)\tan\alpha}{1 - \mu\tan\alpha}\right] = C\,W \qquad (10.10)$$

where C is the lifting coefficient for the wedge acting as a simple machine. It is given by:

$$C = \frac{2\mu + (1 - \mu^2)\tan\alpha}{1 - \mu\tan\alpha} \qquad (10.11)$$

For the wedge to be effective as a machine and lift a heavy weight W with a small force F, we seek to minimize the coefficient C. This minimization is accomplished by reducing the friction coefficient and by using small wedge angles. A graph showing the machine coefficient C as a function of the wedge angle for several different values of the friction coefficient is presented in Fig. 10.15.

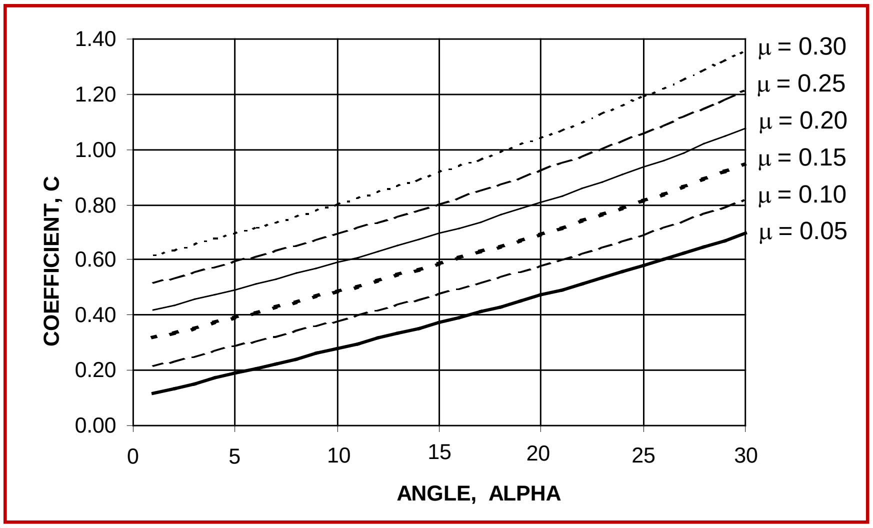


Fig. 10.15 Small forces lift heavy weights with wedges when α is small and μ is low.

EXAMPLE 10.16

You are attempting to level a heavy machine bed with a mass of 45,000 kg. To lift one end of the bed, you insert a wedge with an angle α = 6° under it. If the coefficient of friction between the wedge and the machine bed and the floor is μ = 0.20, determine the force necessary to lift the bed.

Solution:

The FBD of the heavy machine bed as it is lifted with a wedge is presented in Fig E10.16.

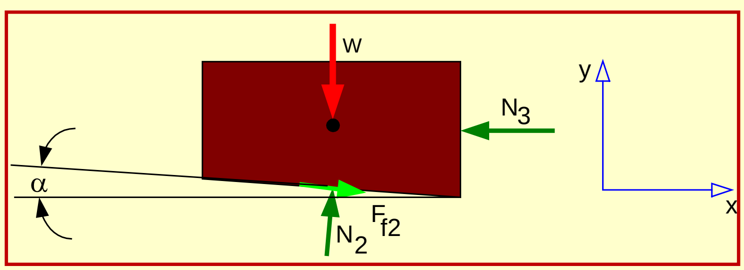


Fig E10.16

Recognize that this FBD is identical to the FBD presented in Fig. 10.14 and that Eqs. (10.10) and (10.11) can be applied to determine the force necessary to lift the machine bed.

The weight of the machine bed is given by:

$$W = (9.807)(45{,}000) = 441.3 \text{ kN} \qquad \text{(a)}$$

Recall Eq. (10.10)

$$F = C\,W = (0.5117)(441.3) = 225.8 \text{ kN} \qquad \text{(b)}$$

where C = 0.5117 was determined from Eq. (10.11).

Suppose that the only hydraulic jack that is available to you has a capacity of 20 tons (177.9 kN). Clearly this jack cannot drive the wedge with the conditions listed above. What can you do to resolve this situation and lift the machine bed?

There are three possible solutions.

- Purchase a larger capacity jack. This option is expensive and will delay the project.
- Find a wedge with a smaller angle. This is a valid solution if a smaller angle wedge exists. If not, ordering a new wedge incurs expense and delays the project.
- Lubricate the wedge with a heavy grease, which will reduce the friction coefficient to $\mu = 0.11$. This is the best solution, because grease is readily available at a low cost.

If we apply a coating of grease to both sides of the wedge, the friction coefficient is reduced to $\mu = 0.11$. Then for a wedge angle of $\alpha = 6°$, the machine coefficient is C = 0.3276. Then:

$$F = C\,W = (0.3276)(441.3 \text{ kN}) = 144.6 \text{ kN} \qquad \text{(c)}$$

This result for F is equivalent to 16.25 tons so the 20-ton capacity jack, which is available, provides sufficient force to drive the wedge.

10.5.2 Holding a Load with a Wedge

Suppose the wedge is driven to raise the weight, as illustrated in Fig. 10.14. The question is then whether or not the wedge will stay in place after the force F is removed. The normal force N_2 has a component tending to push the wedge outward. This force also resists the weight acting downward. Frictional forces must be sufficiently large to prevent the wedge from being ejected. If the friction forces are insufficient, the weight will fall, and the situation could be dangerous.

Let's explore the conditions under which the wedge will hold the load in the absence of the driving force F. As usual, we begin with the free body diagram shown in Fig. 10.16.

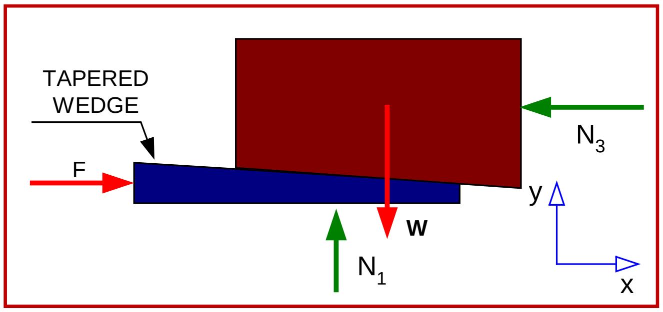


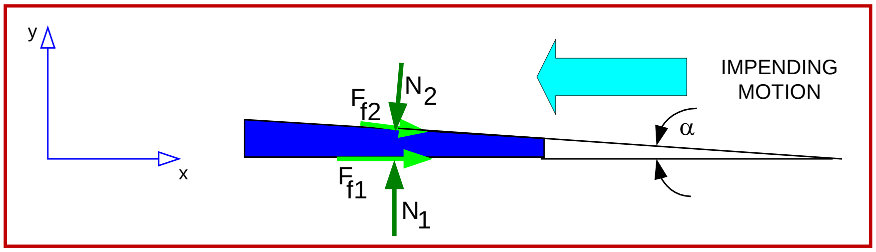
Fig. 10.16 FBD of block and wedge together with a force F applied to the wedge.

For the FBD shown in Fig. 10.16, the equilibrium relation $\Sigma F_y = 0$ yields:

$$\Sigma F_y = 0 \quad \Rightarrow \quad W = N_1 \qquad \text{(a)}$$

Next consider the FBD of the wedge presented in Fig. 10.17.

Fig. 10.17 FBD under holding conditions with no force F applied to the wedge.



For the wedge, we may write:

$$\Sigma F_y = N_1 - N_2 \cos \alpha - F_{f2} \sin \alpha = 0 \qquad \text{(b)}$$

Because the wedge slipping implies impending motion, we may use Eq. (10.3) and write:

$$F_{f1} = \mu N_1 \quad \Rightarrow \quad F_{f2} = \mu N_2 \qquad \text{(c)}$$

$$\Sigma F_x = - N_2 \sin \alpha + F_{f2} \cos \alpha + F_{f1} = 0 \qquad \text{(d)}$$

Combining Eqs. (a) through (d) leads to:

$$\tan \alpha = (2\mu)/(1 - \mu^2) \qquad (10.12)$$

This relation obviously needs some interpretation because the quantities α and μ are usually specified conditions and they are not physically related. We have developed a relation in Eq. (10.12) that describes the condition for the wedge to begin to slip from under the weight. For stability or instability of the weight, you explore the inequalities on both sides of Eq. (10.12), and write:

$$\tan \alpha < (2\mu)/(1 - \mu^2) \qquad (10.13)$$

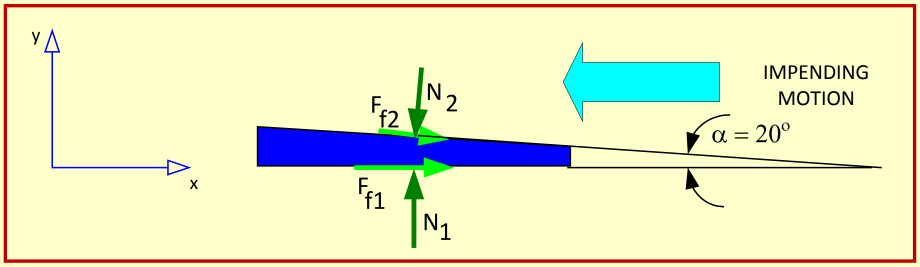
This relation defines the locking angle for a wedge; the wedge remains fixed after it is driven into position if this equation is satisfied. On the other hand, the wedge will slip and the weight will fall (unstable) if:

$$\tan \alpha > (2\mu)/(1 - \mu^2) \qquad (10.14)$$

EXAMPLE 10.17

A wedge with an angle α = 20° is used to lift a 2,500 lb weight. The wedge is lubricated to facilitate the lifting operation, and its coefficient of friction with the two mating surfaces is μ = 0.10. Determine if the wedge will slip when the driving force is removed.

Fig. E10.17



Solution:

The FBD for the wedge with the applied force F removed is shown in Fig. E10.17. We recognize that the stability of the wedge was determined by the analysis leading to Eqs. (10.13) and (10.14). Using these two equations to determine stability of the wedge gives:

$$\tan\alpha < (2\mu)/(1-\mu^2) \quad \Rightarrow \quad \text{stability condition} \qquad (a)$$

$$\tan(20°) = 0.3640 \ \text{ and } \ 2\mu/(1-\mu^2) = (2)(0.10)/[1-(0.10)^2] = 0.2020 \qquad (b)$$

Because $\tan(20°) = 0.3640 > [(2\mu)/(1-\mu^2)] = 0.2020$, Eq. (10.14) is satisfied and an instability condition exists. The wedge will slip and slide out, and the weight will fall.

10.5.3 Removing the Wedge

Suppose that the wedge is a tapered key that is used to fix a gear to a shaft. In some instances, the gear must be removed from the shaft to perform maintenance on the gear, shaft or the bearings. In these cases, a force is applied to pull the wedge from the keyway. Let's determine the force F_R required to pull the key. As usual, we begin with the free body diagram presented in Fig. 10.18 to model the physical situation and to guide us in writing the appropriate relations.

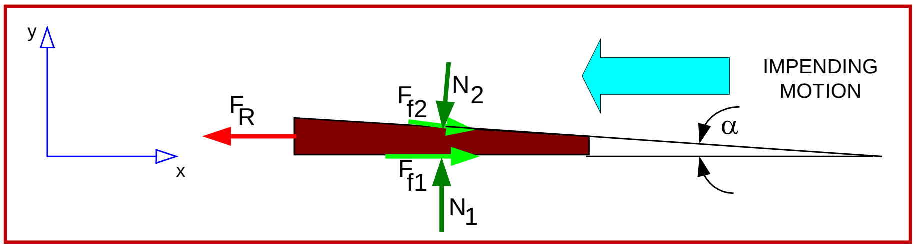
Let's assume the pressure of the gear acting on the wedge-like key produces a force W that is equal to force N_1 as defined in Fig. 10.18.

$$W = N_1 \qquad (a)$$

For the wedge, we may write:

$$\Sigma F_y = N_1 - N_2\cos\alpha - F_{f2}\sin\alpha = 0 \qquad (b)$$

Fig. 10.18 FBD of a wedge removed from a keyway.



Because pulling the wedge implies impending motion, we may use Eq. (10.3) and write:

$$F_{f1} = \mu N_1 \quad \Rightarrow \quad F_{f2} = \mu N_2 \qquad (c)$$

$$\Sigma F_x = -N_2\sin\alpha + F_{f2}\cos\alpha + F_{f1} - F_R = 0 \qquad (d)$$

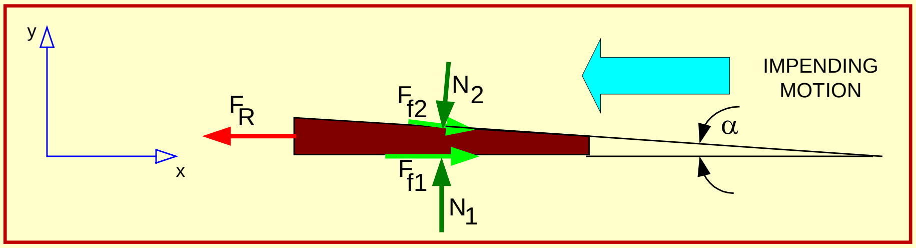
Combining Eqs. (a) through (d) leads to:

$$F_R = \frac{W[2\mu - (1-\mu^2)\tan\alpha]}{1+\mu\tan\alpha} \tag{10.15}$$

EXAMPLE 10.18

Determine the force required to pull a key if the coefficient of friction μ = 0.43 and the angle α of the key is 6°. The pressure between the gear and the key produces a force W of 5,000 lbs.

Fig. E10.18



Solution:

The FBD for the wedge with the force F_R used to pull the key is shown in Fig. E10.18. We recognize that the magnitude of this force was determined by the analysis leading to Eq. (10.15).

Let's employ Eq. (10.15) and write:

$$F_R = \frac{W[2\mu - (1-\mu^2)\tan\alpha]}{1+\mu\tan\alpha} = \frac{5{,}000\{(2)(0.43) - [1-(0.43)^2]\tan(6^\circ)\}}{1+(0.43)\tan(6^\circ)} \tag{a}$$

$$F_R = 3{,}704 \text{ lb} \tag{b}$$

The result shows that very large forces are often required to remove tapered keys from keyways in a disassembly process.

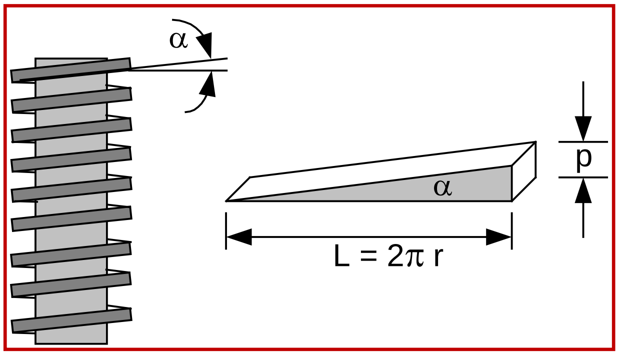
10.6 FRICTION EFFECTS ON SCREWS

Screws are used as fasteners to hold two or more components together in an assembly. They are also used in jacks to lift bodies. In all probability, you have a screw jack in your automobile that is used in lifting one corner of your car when changing a flat tire. Screws are also used in positioning and measuring devices. For example, a screw is used to move the carriage that supports the cutting tool on a lathe, and a screw is used to control the position of the head on a micrometer.

A screw is very much like a wedge or an inclined plane that is wrapped around a cylinder as shown in Fig. 10.19. The inclined plane is wrapped around the cylinder in the form of a helix with an angle α to form the thread. A nut, not shown in Fig. 10.19, is fitted over the thread. As the thread is rotated, the nut is driven in one direction or the other depending on the direction of rotation. In one rotation, the nut is driven a distance of p, which is the pitch of the thread. In driving the carriage of a lathe, the thread rotates in a fixed position and the nut drives the carriage along the length of the lathe bed. In a typical jack, the nut is fixed to the support structure of the jack and the entire screw moves to either lift or lower an automobile.

We apply a moment to turn the screw in either lifting or lowering a load. The frictional forces and the amount of the load determine the magnitude of the moment applied to the screw. Let's determine the magnitude of the moment (torque) required to lift a load.

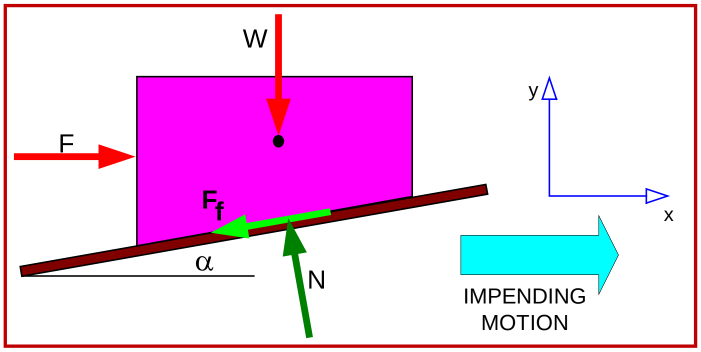
Fig. 10.19 Representing a screw thread with an inclined plane.



10.6.1 Lifting a Load with a Screw

Suppose we are lifting a load W with a screw jack. In designing the jack, we first determine the moment required to lift the load. Let's begin by modeling the thread on the screw with an inclined plane and the nut with a block, as shown in the FBD of Fig. 10.20.

Fig. 10.20 FBD of a model of a screw and nut for lifting a load W.



Apply the equilibrium relations to determine the moment required to lift the load.

$$\Sigma F_y = N \cos \alpha - F_f \sin \alpha - W = 0 \qquad \text{(a)}$$

$$\Sigma F_x = F - N \sin \alpha - F_f \cos \alpha = 0 \qquad \text{(b)}$$

With impending motion it is evident that $F_f = \mu N$. Using this expression in Eq. (a) gives:

$$N = W/(\cos \alpha - \mu \sin \alpha) \qquad \text{(c)}$$

Substituting Eq. (c) and $F_f = \mu N$ into Eq. (b) yields:

$$F = W (\tan \alpha + \mu)/(1 - \mu \tan \alpha) \qquad \text{(d)}$$

However, the relation for the moment M required to turn the screw is:

$$M = F (r) \qquad \text{(10.16)}$$

where r is the radius of the screw.

Then from Eq. (d) and Eq. (10.16), we write:

$$\boxed{M = \frac{Wr(\tan\alpha + \mu)}{1 - \mu\tan\alpha}} \qquad (10.17)$$

This moment M is usually too large for a screw to be turned by hand. When using a screw jack to lift a load, a lever or a wrench with a radius R is often employed. When using a lever the force F required to generate the moment applied to the screw is F = M/R, where M is determined from Eq. (10.17)

10.6.2 Lowering a Load with a Screw

The analysis of the screw employed to lower a load is almost identical to that shown in Section 10.6.1. The only differences are the direction of the frictional force F_f and applied force F, as shown in Fig. 10.21.

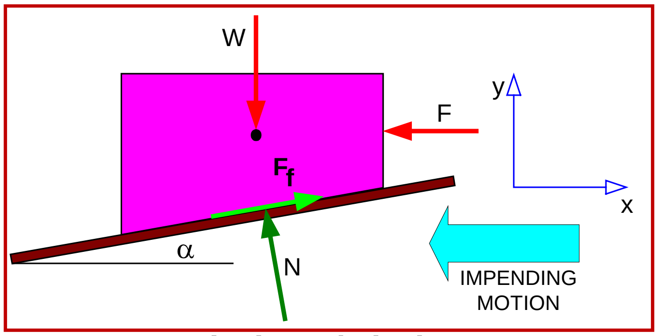


Fig. 10.21 The FBD of a screw and nut used in lowering the load W.

We apply the equilibrium relations to determine the moment required to lower the load.

$$\Sigma F_y = N \cos\alpha + F_f \sin\alpha - W = 0 \qquad (a)$$

$$\Sigma F_x = - F - N \sin\alpha + F_f \cos\alpha = 0 \qquad (b)$$

With impending motion it is evident that $F_f = \mu N$. Using this expression in Eq. (a) gives:

$$N = W/(\cos\alpha + \mu \sin\alpha) \qquad (c)$$

Substituting Eq. (c) and $F_f = \mu N$ into Eq. (b) yields:

$$F = \frac{W(\mu - \tan\alpha)}{1 + \mu\tan\alpha} \qquad (d)$$

Then by substituting Eq. (d) into (10.16), we obtain:

$$\boxed{M = \frac{Wr(\mu - \tan\alpha)}{1 + \mu\tan\alpha}} \qquad (10.18)$$

10.6.3 Holding a Load with a Screw

The analysis of the screw employed to hold the load in a fixed position is similar to the two previous derivations. The friction force F_f opposes the impending motion and the applied force F is absent. We show the FBD for the new model, which represents lowering the load with a screw jack in Fig. 10.22.

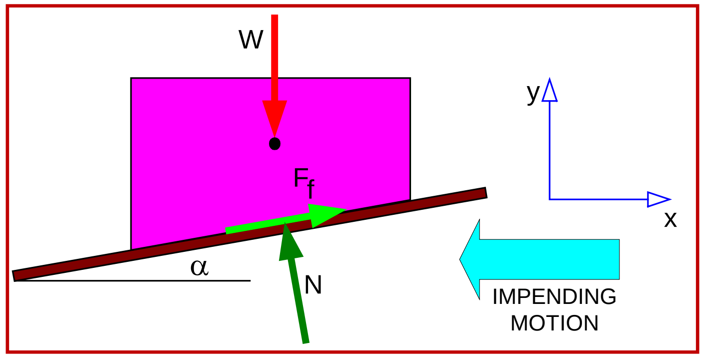


Fig. 10.22 The FBD for a screw holding a load without an applied moment (torque).

Apply the equilibrium relations to determine if the jack will be stable without applying a moment to the screw. This is an important consideration, because self-locking screws are essential for the safe operation of any lifting device. No one wants to have the load crashing down as soon as the moment (torque) is removed from the screw.

Let's begin by writing the equations of equilibrium for the block shown in Fig. 10.22.

$$\Sigma F_x = - N \sin \alpha + F_f \cos \alpha = 0 \qquad \text{(a)}$$

We again assume impending motion and substitute $F_f = \mu N$ into Eq. (a) to obtain:

$$\mu = (\sin \alpha)/(\cos \alpha) = \tan \alpha \qquad \text{(b)}$$

Equation (b) describes the equilibrium condition for the screw with the moment M = 0. However, to insure stability, we write the inequality condition as:

$$\tan \alpha < \mu \qquad (10.19)$$

Substituting Eq. (10.6) into Eq. (10.19) gives:

$$\tan \alpha < \tan \phi \qquad (10.20)$$

Clearly, Eq. (10.20) shows that the helix angle α of the screw must be less than the friction angle ϕ for the screw to be self-locking.

Another method for showing the self-locking condition for a screw is to set M = 0 in Eq. (10.18). Then it is clear that (μ − tan α) = 0 with M = 0, and the conditions for stability given in Eq. (10.19) are obtained.

10.7 <u>SUMMARY</u>

Friction occurs when two or more bodies are in contact and you attempt to move one body relative to the other. Although motion may not occur, friction forces develop that must be considered in any equilibrium analysis of the individual bodies. Friction effects are beneficial, when friction forces maintain stability and provide traction. However, in many instances friction forces are detrimental, because they result in losses of power, energy, and efficiency and because they generate heat.

Friction is due to interactions of the asperities on the contacting surfaces. Under dry conditions, the frictional forces increase until they become sufficiently large to break the bonds between adjoining asperities and/or shear the interlocking asperities. Lubricating the surfaces markedly reduces the frictional forces, because the lubricants physically separate the asperities. The small friction forces developed under fully lubricated conditions are due to shearing the fluid lubricant.

The magnitude of the friction force is variable. It increases up to some maximum value to prevent relative motion between the contacting surfaces. The maximum friction force is related to the shearing force necessary to destroy the bonds between the contacting asperities. It is approximated by:

$$(F_f)_{Max} = \mu N \qquad (10.3)$$

Table 10.1 provides approximate values for the coefficients of friction μ for different material pairs; however, typical friction coefficients exhibit large variations. Hence, it is advisable to measure μ with a simple experiment involving an inclined plane, as illustrated in Fig. 10.7.

There is a difference between the static and dynamic coefficients of friction. The static coefficient of friction is associated with the maximum friction force that may be developed before motion occurs. The dynamic coefficient is associated with a constant value of the friction force after motion begins. The dynamic coefficient is usually 20 to 25% lower than the static coefficient.

We often specify the coefficient of friction with either the friction angle or the angle of repose. These angles are equal. The angle of repose θ_s is measured with an inclined plane, and the friction angle ϕ is the angle between the resultant force and the friction force, as shown in Fig. 10.9. The tangent of both of these angles is equal to the coefficient of friction.

$$\tan \theta_s = \tan \phi = \mu \qquad (10.5 \ \& \ 10.6)$$

Many examples are presented to illustrate procedures useful in solving "friction" problems. These procedures include:

- Construction of a FBD to model the physical situation and to show all of the forces acting on the body and the dimensions of the model when necessary.
- Application of the appropriate equations of equilibrium.
- Execution of the mathematics required to solve these equations.

Friction dependent stability was demonstrated by considering a number of different examples involving the stability of a ladder. We assume a condition of impending motion and derive an inequality relation that indicates stability or the lack thereof:

$$\begin{aligned} F_f &\leq (F_f)_{Max} \quad \Rightarrow \quad \text{for stability.} \\ F_f &> (F_f)_{Max} \quad \Rightarrow \quad \text{for instability.} \end{aligned} \qquad \text{(a)}$$

Examples showing the role of friction in tipping versus sliding were considered. The determination involves locating the point of application of the normal reactive force. If this normal force is positioned to the right of point A in Fig. 10.10, the crate tips rather than slides.

An example was introduced to show the conditions for rolling or slipping. The determination of whether a cylinder rolls or slips is identical to the stability condition for the ladder.

We examined the role of friction relative to both the wedge and the screw, which are simple machines. In both components, three situations must be analyzed — lifting, holding and lowering. All of the equations necessary to determine the driving force and or moment (torque) required to perform these three functions for both the wedge and the screw are summarized below:

The force required to lift a weight W with a wedge is:

$$F = W\left[\frac{2\mu + (1 - \mu^2)\tan\alpha}{1 - \mu\tan\alpha}\right] = C\,W \qquad (10.10)$$

$$C = \frac{2\mu + (1 - \mu^2)\tan\alpha}{1 - \mu\tan\alpha} \qquad (10.11)$$

With no force applied a wedge will hold (stable) if the inequality below is satisfied:

$$\tan\alpha < 2\mu/(1 - \mu^2) \qquad (10.13)$$

On the other hand the wedge will slip, eject and the weight will fall (unstable) if:

$$\tan\alpha > 2\mu/(1 - \mu^2) \qquad (10.14)$$

The force F_R required to remove a wedge is given by:

$$F_R = \frac{W[2\mu - (1 - \mu^2)\tan\alpha]}{1 + \mu\tan\alpha} \qquad (10.15)$$

The moment that must be applied to a screw to lift a load W is given by:

$$M = \frac{Wr(\tan\alpha + \mu)}{1 - \mu\tan\alpha} \qquad (10.17)$$

The moment that must be applied to a screw to lower a load W is given by:

$$M = \frac{Wr(\mu - \tan\alpha)}{1 + \mu\tan\alpha} \qquad (10.18)$$

Finally the condition for stability of a screw in holding a load is given by:

$$\tan\alpha < \tan\phi = \mu \qquad (10.20)$$

The conditions for holding a wedge or screw in place when the lifting force or moment is removed are particularly important. If a screw or wedge does not hold when the applied force or moment is removed, the machine is unstable. The weight will fall and injury to persons or damage to property may result.

APPENDICES

APPENDIX A

WIRE AND SHEET METAL GAGES (Dimensions are in inches)

Gage No. of Wire	B & S[1] Non Ferrous Metals	American S. & W Steel Wire	American S. & W Music Wire	Steel Manufactures' Sheet	Gage No. of Wire
7-0s	0.651354	0.4900	…...	……	7-0s
6-0s	0.580049	0.4615	0.004	……	6-0s
5-0s	0.516549	0.4305	0.005	……	5-0s
4-0s	0.4600	0.3938	0.006	……	4-0s
000s	0.40964	0.3625	0.007	……	000s
00	0.3648	0.3310	0.008	……	00
0	0.32486	0.3065	0.009	……	0
1	0.2893	0.2830	0.010	……	1
2	0.25763	0.2625	0.011	……	2
3	0.22942	0.2437	0.012	0.2391	3
4	0.20431	0.2253	0.013	0.2242	4
5	0.18194	0.2070	0.014	0.2092	5
6	0.16202	0.1920	0.016	0.1943	6
7	0.14428	0.1770	0.018	0.1793	7
8	0.12849	0.1620	0.020	0.1644	8
9	0.11443	0.1483	0.022	0.1495	9
10	0.10189	0.1350	0.024	0.1345	10
11	0.090742	0.1205	0.026	0.1196	11
12	0.080808	0.1055	0.029	0.1046	12
13	0.071961	0.0915	0.031	0.0897	13
14	0.064084	0.0800	0.033	0.0747	14
15	0.057068	0.0720	0.035	0.0763	15
16	0.05082	0.0625	0.037	0.0598	16
17	0.045257	0.0540	0.039	0.0538	17
18	0.040303	0.0475	0.041	0.0478	18
19	0.03589	0.0410	0.043	0.0418	19
20	0.031961	0.0348	0.045	0.0359	20
21	0.028462	0.0317	0.047	0.0329	21
22	0.025347	0.0286	0.049	0.0299	22
23	0.022571	0.0258	0.051	0.0269	23
24	0.02010	0.0230	0.055	0.0239	24
25	0.01790	0.0204	0.059	0.0209	25
26	0.01594	0.0181	0.063	0.0179	26
27	0.014195	0.0173	0.067	0.0164	27
28	0.012641	0.0162	0.071	0.0149	28
29	0.011257	0.0150	0.075	0.0135	29
30	0.010025	0.0140	0.080	0.0120	30
31	0.008928	0.0132	0.085	0.0105	31
32	0.00795	0.0128	0.090	0.0097	32
33	0.00708	0.0118	0.095	0.0090	33
34	0.006304	0.0104	……	0.0082	34
35	0.005614	0.0095	……	0.0075	35
36	0.00500	0.0090	……	0.0067	36
37	0.004453	0.0085	……	0.0064	37
38	0.003965	0.0080	……	0.0060	38
39	0.003531	0.0075	……	……	39
40	0.003144	0.0070	……	……	40

[1] Courtesy of Brown and Sharpe Manufacturing Company.

APPENDIX B-1

PHYSICAL PROPERTIES OF COMMON STRUCTURAL MATERIALS

MATERIAL	ELASTIC MODULUS, E		SHEAR MODULUS, G		POISSON'S RATIO, ν	THERMAL EXPANSION COEFFICIENT, α	
	Mpsi	GPa	Mpsi	GPa	—	$\times 10^{-6}$/°F	$\times 10^{-6}$/°C
METAL							
Aluminum Alloy	10.4	72	3.9	27	0.32	12.9	23.2
Brass, Bronze	16	110	6.0	41	0.33	11.1	20.0
Copper	17.5	121	6.6	46	0.33	9.4	16.9
Cast Iron - Gray	15	103	6.0	41	0.26	6.7	12.1
Cast Iron - Malleable	25	170	9.9	68	0.26	6.7	12.1
Magnesium Alloy	6.5	45	2.4	17	0.35	14.4	25.9
Nickel Alloy	30	207	11.5	79	0.30	7.8	14.0
Steel	30	207	11.5	79	0.30	6.3	11.3
Stainless Steel	27.5	190	10.6	73	0.30	9.6	17.3
Titanium Alloy	16.5	114	6.2	43	0.33	4.9	8.8
WOOD[1]							
Douglas Fir	1.9	13	0.1	0.7	—	2.1	3.8
Sitka Spruce	1.5	10	0.07	0.5	—	2.1	3.8
Western White Pine	1.5	10	—	—	—	2.1	3.8
White Oak	1.8	12	—	—	—	2.1	3.8
Red Oak	1.8	12	—	—	—	2.1	3.8
Redwood	1.3	9	—	—	—	2.1	3.8
CONCRETE							
Medium Strength	3.6	25	—	—	—	5.5	9.9
High Strength	4.5	31	—	—	—	5.5	9.9
PLASTIC							
Nylon Type 6/6	0.4	2.8	—	—	—	80	144
Polycarbonate	0.35	2.4	—	—	—	68	122
Polyester, PBT	0.35	2.4	—	—	—	75	135
Polystyrene	0.45	3.1	—	—	—	70	125
Vinyl, Rigid PVC	0.45	3.1	—	—	—	75	135
STONE							
Granite	10	70	4	28	0.25	4	7.2
Marble	8	55	3	21	0.33	6	10.8
Sandstone	6	40	2	14	0.50	5	9.0
GLASS	9.6	65	4.1	28	0.17	44	80
RUBBER	0.22[2]	0.0015	0.073[2]	0.0005	0.50	125	225

The values for the properties given above are representative. Because processing methods and exact composition of the material influence the properties to some degree, the exact values may differ from those presented here.

1. Wood is an orthotropic material with different properties in different directions. The values given here are parallel to the grain.
2. The modulus for rubber is given in ksi.

APPENDIX B-2

TENSILE PROPERTIES OF COMMON STRUCTURAL MATERIALS

MATERIAL	ULTIMATE TENSILE STRENGTH, S_u		YIELD STRENGTH, S_y		DENSITY, ρ	
	ksi	MPa	ksi	MPa	lb/in.^{3}	Mg/m^3
CARBON & ALLOY STEELS						
1010 A	44	303	29	200	0.284	7.87
1018 A	49.5	341	32	221	0.284	7.87
1020 HR	66	455	42	290	0.284	7.87
1045 HR	92.5	638	60	414	0.284	7.87
1212 HR	61.5	424	28	193	0.284	7.87
4340 HR	151	1041	132	910	0.283	7.84
52100 A	167	1151	131	903	0.284	7.87
STAINLESS STEELS						
302 A	92	634	34	234	0.286	7.92
303 A	87	600	35	241	0.286	7.92
304 A	83	572	40	276	0.286	7.92
440C A	117	807	67	462	0.286	7.92
CAST IRON						
Gray	25	170	—	—	0.260	7.20
Malleable	50	340	32	220	0.266	7.37
ALUMINUM ALLOYS						
1100-0	12	83	4.5	31	0.098	2.71
2024-T4	65	448	43	296	0.100	2.77
6061-T6	38	260	35	240	0.098	2.71
7075-0	34	234	14.3	99	0.100	2.77
7075 T6	86	593	78	538	0.100	2.77
MAGNESIUM ALLOYS						
HK31XA-0	25.5	176	19	131	0.066	1.83
HK31XA-H24	36.2	250	31	214	0.066	1.83
NICKEL ALLOYS						
Monel 400 A	80	550	32	220	0.319	8.83
Cupronickel A	53	365	16	110	0.323	8.94
COPPER ALLOYS						
Oxygen-free (99.9%) A	32	220	10	70	0.322	8.91
90-10 Brass A	36.4	251	8.4	58	0.316	8.75
80-20 Brass A	35.8	247	7.2	50	0.316	8.75
70-30 Brass A	44.0	303	10.5	72	0.316	8.75
Naval Brass	54.5	376	17	117	0.316	8.75
Tin Bronze	45	310	21	145	0.318	8.80
Aluminum Bronze	90	620	40	275	0.301	8.33
TITANIUM ALLOY						
Annealed	155	1070	135	930	0.167	4.63

A = Annealed and HR = Hot Rolled

APPENDIX B-3

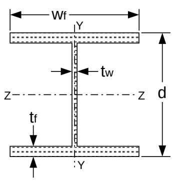
TENSILE PROPERTIES OF NON-METALLIC MATERIALS

MATERIAL	ULTIMATE TENSILE STRENGTH, S_u		YIELD STRENGTH, S_y		DENSITY, ρ	
	ksi	MPa	ksi	MPa	lb/in.^{3}	Mg/m^3
WOOD						
Douglas Fir	15	100	—	—	0.017	0.470
Sitka Spruce	8.6	60	—	—	0.015	0.415
Western White Pine	5.0	34	—	—	0.014	0.390
White Oak	7.4	51	—	—	0.025	0.690
Red Oak	6.8	47	—	—	0.024	0.660
Redwood	9.4	65	—	—	0.015	0.415
CONCRETE[1]						
Medium Strength	4.0	28	—	—	0.084	2.32
High Strength	6.0	40	—	—	0.084	2.32
PLASTIC						
Nylon Type 6/6	11	75	6.5	45	0.0412	1.14
Polycarbonate	9.5	65	9	62	0.0433	1.20
Polyester, PBT	8	55	8	55	0.0484	1.34
Polystyrene	8	55	8	55	0.0374	1.03
Vinyl, Rigid PVC	6.5	45	6	40	0.0520	1.44
STONE[1]						
Granite	35	240	—	—	0.100	2.77
Marble	18	125	—	—	0.100	2.77
Sandstone	12	85	—	—	0.083	2.30
GLASS[1]						
98% Silica	7	50	—	—	0.079	2.19
RUBBER						
Natural, Vulcanized	4	28	—	—	0.034	0.95

1. The tensile strength of concrete, stone and bulk glass is negligible. The compressive strength for these materials is reported in this table.

APPENDIX C

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES

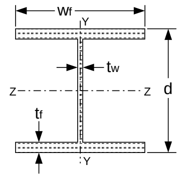


Wide-Flange Beams (U.S. Customary Units)

Designation*	Area ($in.^2$)	Depth (in.)	Flange Width (in.)	Flange Thickness (in.)	Web Thickness (in.)	Axis Z-Z I ($in.^4$)	Axis Z-Z Z ($in.^3$)	Axis Z-Z r (in.)	Axis Y-Y I ($in.^4$)	Axis Y-Y Z ($in.^3$)	Axis Y-Y r (in.)
W36 × 230	67.6	35.90	16.470	1.260	0.760	15000	837	14.9	940	114	3.73
× 160	47.0	36.01	12.000	1.020	0.650	9750	542	14.4	295	49.1	2.50
W33 × 201	59.1	33.68	15.745	1.150	0.715	11500	684	14.0	749	95.2	3.56
× 152	44.7	33.49	11.565	1.055	0.635	8160	487	13.5	273	47.2	2.47
× 130	38.3	33.09	11.510	0.855	0.580	6710	406	13.2	218	37.9	2.39
W30 × 132	38.9	30.31	10.545	1.000	0.615	5770	380	12.2	196	37.2	2.25
× 108	31.7	29.83	10.475	0.760	0.545	4470	299	11.9	146	27.9	2.15
W27 × 146	42.9	27.38	13.965	0.975	0.605	5630	411	11.4	443	63.5	3.21
× 94	27.7	26.92	9.990	0.745	0.490	3270	243	10.9	124	24.8	2.12
W24 × 104	30.6	24.06	12.750	0.750	0.500	3100	258	10.1	259	40.7	2.91
× 84	24.7	24.10	9.020	0.770	0.470	2370	196	9.79	94.4	20.9	1.95
× 62	18.2	23.74	7.040	0.590	0.430	1550	131	9.23	34.5	9.80	1.38
W21 × 101	29.8	21.36	12.290	0.800	0.500	2420	227	9.02	248	40.3	2.89
× 83	24.3	21.43	8.355	0.835	0.515	1830	171	8.67	81.4	19.5	1.83
× 62	18.3	20.99	8.240	0.615	0.400	1330	127	8.54	57.5	13.9	1.77
W18 × 97	28.5	18.59	11.145	0.870	0.535	1750	188	7.82	201	36.1	2.65
× 76	22.3	18.21	11.035	0.680	0.425	1330	146	7.73	152	27.6	2.61
× 60	17.6	18.24	7.555	0.695	0.415	984	108	7.47	50.1	13.3	1.69
W16 × 100	29.4	16.97	10.425	0.985	0.585	1490	175	7.10	186	35.7	2.52
× 67	19.7	16.33	10.235	0.665	0.395	954	117	6.96	119	23.2	2.46
× 40	11.8	16.01	6.995	0.505	0.305	518	64.7	6.63	28.9	8.25	1.57
× 26	7.68	15.69	5.500	0.345	0.250	301	38.4	6.26	9.59	3.49	1.12
W14 × 120	35.3	14.48	14.670	0.940	0.590	1380	190	6.24	495	67.5	3.74
× 82	24.1	14.31	10.130	0.855	0.510	882	123	6.05	148	29.3	2.48
× 43	12.6	13.66	7.995	0.530	0.305	428	62.7	5.82	45.2	11.3	1.89
× 30	8.85	13.84	6.730	0.385	0.270	291	42.0	5.73	19.6	5.82	1.49
W12 × 96	28.2	12.71	12.160	0.900	0.550	833	131	5.44	270	44.4	3.09
× 65	19.1	12.12	12.000	0.605	0.390	533	87.9	5.28	174	29.1	3.02
× 50	14.7	12.19	8.080	0.640	0.370	394	64.7	5.18	56.3	13.9	1.96
× 30	8.79	12.34	6.520	0.440	0.260	238	38.6	5.21	20.3	6.24	1.52
W10 × 60	17.6	10.22	10.080	0.680	0.420	341	66.7	4.39	116	23.0	2.57
× 45	13.3	l0.l0	8.020	0.620	0.350	248	49.1	4.33	53.4	13.3	2.01
× 30	8.84	10.47	5.810	0.510	0.300	170	32.4	4.38	16.7	5.75	1.37
× 22	6.49	10.17	5.750	0.360	0.240	118	23.2	4.27	11.4	3.97	1.33
W8 × 40	l 1.7	8.25	8.070	0.560	0.360	146	35.5	3.53	49.1	12.2	2.04
× 31	9.13	8.00	7.995	0.435	0.285	1l0	27.5	3.47	37.1	9.27	2.02
× 24	7.08	7.93	6.495	0.400	0.245	82.8	20.9	3.42	18.3	5.63	1.61
× 15	4.44	8.11	4.015	0.315	0.245	48.0	11.8	3.29	3.41	1.70	0.876
W6 × 25	7.34	6.38	6.080	0.455	0.320	53.4	16.7	2.70	17.1	5.61	1.52
× 16	4.74	6.28	4.030	0.405	0.260	32.1	10.2	2.60	4.43	2.20	0.967
W5 × 16	4.68	5.01	5.000	0.360	0.240	21.3	8.51	2.13	7.51	3.00	1.27
W4 × 13	3.83	4.16	4.060	0.345	0.280	11.3	5.46	1.72	3.86	1.90	1.00

Courtesy of the American Institute of Steel Construction.
*W is the symbol for a wide-flange beam, followed by the nominal depth in inches, and weight in pounds per foot of length.

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES

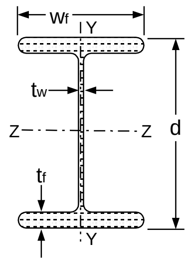


Wide-Flange Beams (SI Units)

Designation*	Area (mm^2)	Depth (mm)	Width (mm)	Flange Thickness (mm)	Web Thickness (mm)	Axis Z-Z I (10^6 mm^4)	Axis Z-Z Z (10^3 mm^3)	Axis Z-Z r (mm)	Axis Y-Y I (10^6 mm^4)	Axis Y-Y Z (10^3 mm^3)	Axis Y-Y r (mm)
W914 × 342	43610	912	418	32.0	19.3	6245	13715	378	391	1870	94.7
× 238	30325	915	305	25.9	16.5	4060	8880	366	123	805	63.5
W838 × 299	38130	855	400	29.2	18.2	4785	11210	356	312	1560	90.4
× 226	28850	851	294	26.8	16.1	3395	7980	343	114	775	62.7
× 193	24710	840	292	21.7	14.7	2795	6655	335	90.7	620	60.7
W762 × 196	25100	770	268	25.4	15.6	2400	6225	310	81.6	610	57.2
× 161	20450	758	266	19.3	13.8	1860	4900	302	60.8	457	54.6
W686 × 217	27675	695	355	24.8	15.4	2345	6735	290	184	1040	81.5
× 140	17870	684	254	18.9	12.4	1360	3980	277	51.6	406	53.8
W610 × 155	19740	611	324	19.1	12.7	1290	4230	257	108	667	73.9
× 125	15935	612	229	19.6	11.9	985	3210	249	39.3	342	49.5
× 92	11750	603	179	15.0	10.9	645	2145	234	14.4	161	35.1
W533 × 150	19225	543	312	20.3	12.7	1005	3720	229	103	660	73.4
× 124	15675	544	212	21.2	13.1	762	2800	220	33.9	320	46.5
× 92	11805	533	209	15.6	10.2	554	2080	217	23.9	228	45.0
W457 × 144	18365	472	283	22.1	13.6	728	3080	199	83.7	592	67.3
× 113	14385	463	280	17.3	10.8	554	2395	196	63.3	452	66.3
× 89	11355	463	192	17.7	10.5	410	1770	190	20.9	218	42.9
W406 × 149	18970	431	265	25.0	14.9	620	2870	180	77.4	585	64.0
× 100	12710	415	260	16.9	10.0	397	1915	177	49.5	380	62.5
× 60	7615	407	178	12.8	7.7	216	1060	168	12.0	135	39.9
× 39	4950	399	140	8.8	6.4	125	629	159	3.99	57.2	28.4
W356 × 179	22775	368	373	23.9	15.0	574	3115	158	206	1105	95.0
× 122	15550	363	257	21.7	13.0	367	2015	154	61.6	480	63.0
× 64	8130	347	203	13.5	7.7	178	1025	148	18.8	185	48.0
× 45	5710	352	171	9.8	6.9	121	688	146	8.16	95.4	37.8
W305 × 143	18195	323	309	22.9	14.0	347	2145	138	112	728	78.5
× 97	12325	308	305	15.4	9.9	222	1440	134	72.4	477	76.7
× 74	9485	310	205	16.3	9.4	164	1060	132	23.4	228	49.8
× 45	5670	313	166	11.2	6.6	99.1	633	132	8.45	102	38.6
W254 × 89	11355	260	256	17.3	10.7	142	1095	112	48.3	377	65.3
× 67	8580	257	204	15.7	8.9	103	805	110	22.2	218	51.1
× 45	5705	266	148	13.0	7.6	70.8	531	111	6.95	94.2	34.8
× 33	4185	258	146	9.1	6.1	49.1	380	108	4.75	65.1	33.8
W203 × 60	7550	210	205	14.2	9.1	60.8	582	89.7	20.4	200	51.8
× 46	5890	203	203	11.0	7.2	45.8	451	88.1	15.4	152	51.3
× 36	4570	201	165	10.2	6.2	34.5	342	86.7	7.61	92.3	40.9
× 22	2865	206	102	8.0	6.2	20.0	193	83.6	1.42	27.9	22.3
W152 × 37	4735	162	154	11.6	8.1	22.2	274	68.6	7.12	91.9	38.6
× 24	3060	160	102	10.3	6.6	13.4	167	66.0	1.84	36.1	24.6
W127 × 24	3020	127	127	9.1	6.1	8.87	139	54.1	3.13	49.2	32.3
W102 × 19	2470	106	103	8.8	7.1	4.70	89.5	43.7	1.61	31.1	25.4

Courtesy of The American Institute of Steel Construction.
*W is the symbol for a wide-flange beam, followed by the nominal depth in mm, and the mass in kg per meter of length.

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES

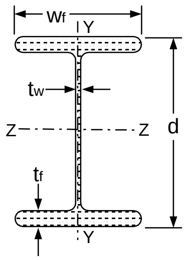


American Standard Beams (U.S. Customary Units)

Designation*	Area (in^2)	Depth (in.)	Width (in.)	Flange Thickness (in.)	Web Thickness (in.)	Axis Z-Z I ($in.^4$)	Axis Z-Z Z ($in.^3$)	Axis Z-Z r (in.)	Axis Y-Y I ($in.^4$)	Axis Y-Y Z ($in.^3$)	Axis Y-Y r (in.)
S24 × 121	35.6	24.50	8.050	1.090	0.800	3160	258	9.43	83.3	20.7	1.53
× 106	31.2	24.50	7.870	1.090	0.620	2940	240	9.71	77.1	19.6	1.57
× 100	29.3	24.00	7.245	0.870	0.745	2390	199	9.02	47.7	13.2	1.27
× 90	26.5	24.00	7.125	0.870	0.625	2250	187	9.21	44.9	12.6	1.30
× 80	23.5	24.00	7.000	0.870	0.500	2100	175	9.47	42.2	12.1	1.34
S20 × 96	28.2	20.30	7.200	0.920	0.800	1670	165	7.71	50.2	13.9	1.33
× 86	25.3	20.30	7.060	0.920	0.660	1580	155	7.89	46.8	13.3	1.36
× 75	22.0	20.00	6.385	0.795	0.635	1280	128	7.62	29.8	9.32	1.16
× 66	19.4	20.00	6.255	0.795	0.505	1190	119	7.83	27.7	8.85	1.19
S18 × 70	20.6	18.00	6.251	0.691	0.711	926	103	6.71	24.1	7.72	1.08
× 54.7	16.1	18.00	6.001	0.691	0.461	804	89.4	7.07	20.8	6.94	1.14
S15 × 50	14.7	15.00	5.640	0.622	0.550	486	64.8	5.75	15.7	5.57	1.03
× 42.9	12.6	15.00	5.501	0.622	0.411	447	59.6	5.95	14.4	5.23	1.07
S12 × 50	14.7	12.00	5.477	0.659	0.687	305	50.8	4.55	15.7	5.74	1.03
× 40.8	12.0	12.00	5.252	0.659	0.462	272	45.4	4.77	13.6	5.16	1.06
× 35	10.3	12.00	5.078	0.544	0.428	229	38.2	4.72	9.87	3.89	0.980
× 31.8	9.35	12.00	5.000	0.544	0.350	218	36.4	4.83	9.36	3.74	1.000
S10 × 35	10.3	10.00	4.944	0.491	0.594	147	29.4	3.78	8.36	3.38	0.901
× 25.4	7.46	10.00	4.661	0.491	0.311	124	24.7	4.07	6.79	2.91	0.954
S8 × 23	6.77	8.00	4.171	0.426	0.441	64.9	16.2	3.10	4.31	2.07	0.798
× 18.4	5.41	8.00	4.001	0.426	0.271	57.6	14.4	3.26	3.73	1.86	0.831
S7 × 20	5.88	7.00	3.860	0.392	0.450	42.4	12.1	2.69	3.17	1.64	0.734
× 15.3	4.50	7.00	3.662	0.392	0.252	36.7	10.5	2.86	2.64	1.44	0.766
S6 × 17.25	5.07	6.00	3.565	0.359	0.465	26.3	8.77	2.28	2.31	1.30	0.675
× 12.5	3.67	6.00	3.332	0.359	0.232	22.1	7.37	2.45	1.82	1.09	0.705
S5 × 14.75	4.34	5.00	3.284	0.326	0.494	15.2	6.09	1.87	1.67	1.01	0.620
× 10	2.94	5.00	3.004	0.326	0.214	12.3	4.92	2.05	1.22	0.809	0.643
S4 × 9.5	2.79	4.00	2.796	0.293	0.326	6.79	3.39	1.56	0.903	0.646	0.569
× 7.7	2.26	4.00	2.663	0.293	0.193	6.08	3.04	1.64	0.764	0.574	0.581
S3 × 7.5	2.21	3.00	2.509	0.260	0.349	2.93	1.95	1.15	0.586	0.468	0.516
× 5.7	1.67	3.00	2.330	0.260	0.170	2.52	1.68	1.23	0.455	0.390	0.522

Courtesy of The American Institute of Steel Construction.
*S is the symbol for a standard beam, followed by the nominal depth in inches, then the weight in pounds per foot of length.

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES

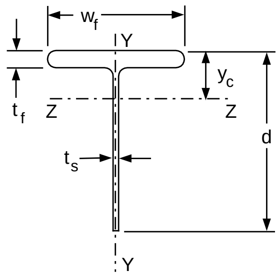


American Standard Beams (SI Units)

Designation*	Area (mm^2)	Depth (mm)	Flange Width (mm)	Flange Thickness (mm)	Web Thickness (mm)	Axis Z-Z I (10^6 mm^4)	Axis Z-Z Z (10^3 mm^3)	Axis Z-Z r (mm)	Axis Y-Y I (10^6 mm^4)	Axis Y-Y Z (10^3 mm^3)	Axis Y-Y r (mm)
S610 × 180	22970	622.3	204.5	27.7	20.3	1315	4225	240	34.7	339	38.9
× 158	20130	622.3	199.9	27.7	15.7	1225	3935	247	32.1	321	39.9
× 149	18900	609.6	184.0	22.1	18.9	995	3260	229	19.9	216	32.3
× 134	17100	609.6	181.0	22.1	15.9	937	3065	234	18.7	206	33.0
× 119	15160	609.6	177.8	22.1	12.7	874	2870	241	17.6	198	34.0
S508 × 143	18190	515.6	182.9	23.4	20.3	695	2705	196	20.9	228	33.8
× 128	16320	515.6	179.3	23.4	16.8	658	2540	200	19.5	218	34.5
× 112	14190	508.0	162.2	20.2	16.1	533	2100	194	12.4	153	29.5
× 98	12520	508.0	158.9	20.2	12.8	495	1950	199	11.5	145	30.2
S457 × 104	13290	457.2	158.8	17.6	18.1	358	1690	170	10.0	127	27.4
× 81	10390	457.2	152.4	17.6	11.7	335	1465	180	8.66	114	29.0
S381 × 74	9485	381.0	143.3	15.8	14.0	202	1060	146	6.53	91.3	26.2
× 64	8130	381.0	139.7	15.8	10.4	186	977	151	5.99	85.7	27.2
S305 × 74	9485	304.8	139.1	16.7	17.4	127	832	116	6.53	94.1	26.2
× 61	7740	304.8	133.4	16.7	11.7	113	744	121	5.66	84.6	26.9
× 52	6645	304.8	129.0	13.8	10.9	95.3	626	120	4.11	63.7	24.1
× 47	6030	304.8	127.0	13.8	8.9	90.7	596	123	3.90	61.3	25.4
S254 × 52	6645	254.0	125.6	12.5	15.1	61.2	482	96.0	3.48	55.4	22.9
× 38	4815	254.0	118.4	12.5	7.9	51.6	408	103	2.83	47.7	24.2
S203 × 34	4370	203.2	105.9	10.8	11.2	27.0	265	78.7	1.79	33.9	20.3
× 27	3490	203.2	101.6	10.8	6.9	24.0	236	82.8	1.55	30.5	21.1
S178 × 30	3795	177.8	98.0	10.0	11.4	17.6	198	68.3	1.32	26.9	18.6
× 23	2905	177.8	93.0	10.0	6.4	15.3	172	72.6	1.10	23.6	19.5
S152 × 26	3270	152.4	90.6	9.1	11.8	10.9	144	57.9	0.961	21.3	17.1
× 19	2370	152.4	84.6	9.1	5.9	9.20	121	62.2	0.758	17.9	17.9
S127 × 22	2800	127.0	83.4	8.3	12.5	6.33	99.8	47.5	0.695	16.6	15.7
× 15	1895	127.0	76.3	8.3	5.4	5.12	80.6	52.1	0.508	13.3	16.3
S102 × 14	1800	101.6	71.0	7.4	8.3	2.83	55.6	39.6	0.376	10.6	14.5
× 11	1460	101.6	67.6	7.4	4.9	2.53	49.8	41.7	0.318	9.41	14.8
S76 × 11	1425	76.2	63.7	6.6	8.9	1.22	32.0	29.2	0.244	7.67	13.1
× 8.5	1075	76.2	59.2	6.6	4.3	1.05	27.5	31.2	0.189	6.39	13.3

Courtesy of The American Institute of Steel Construction.
*S is the symbol for a standard beam, followed by the nominal depth in mm, then the mass in kg per meter of length.

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES

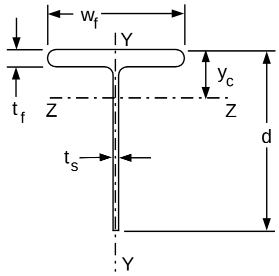


Structural Tees (U.S. Customary Units)

Designation*	Area (in.^{2})	Depth of Tee (in.)	Flange Width (in.)	Flange Thickness (in.)	Stem Thickness (in.)	Axis Z-Z I (in.^{4})	Axis Z-Z Z (in.^{3})	Axis Z-Z r (in.)	y_c (in.)	Axis Y-Y I (in.^{4})	Axis Y-Y Z (in.^{3})	Axis Y-Y r (in.)
WT18 × 115	33.8	17.950	16.470	1.260	0.760	934	67.0	5.25	4.01	470	57.1	3.73
× 80	23.5	18.005	12.000	1.020	0.650	740	55.8	5.61	4.74	147	24.6	2.50
WT15 × 66	19.4	15.155	10.545	1.000	0.615	421	37.4	4.66	3.90	98.0	18.6	2.25
× 54	15.9	14.915	10.475	0.760	0.545	349	32.0	4.69	4.01	73.0	13.9	2.15
WT12 × 52	15.3	12.030	12.750	0.750	0.500	189	20.0	3.51	2.59	130	20.3	2.91
× 47	13.8	12.155	9.065	0.875	0.515	186	20.3	3.67	2.99	54.5	12.0	1.98
× 42	12.4	12.050	9.020	0.770	0.470	166	18.3	3.67	2.97	47.2	10.5	1.95
× 31	9.11	11.870	7.040	0.590	0.430	131	15.6	3.79	3.46	17.2	4.90	1.38
WT9 × 38	11.2	9.105	11.035	0.680	0.425	71.8	9.83	2.54	1.80	76.2	13.8	2.61
× 30	8.82	9.120	7.555	0.695	0.415	64.7	9.29	2.71	2.16	25.0	6.63	1.69
× 25	7.33	8.995	7.495	0.570	0.355	53.5	7.79	2.70	2.12	20.0	5.35	1.65
× 20	5.88	8.950	6.015	0.525	0.315	44.8	6.73	2.76	2.29	9.55	3.17	1.27
WT8 × 50	14.7	8.485	10.425	0.985	0.585	76.8	11.4	2.28	1.76	93.1	17.9	2.51
× 25	7.37	8.130	7.070	0.630	0.380	42.3	6.78	2.40	1.89	18.6	5.26	1.59
× 20	5.89	8.005	6.995	0.505	0.305	33.1	5.35	2.37	1.81	14.4	4.12	1.57
× 13	3.84	7.845	5.500	0.345	0.250	23.5	4.09	2.47	2.09	4.80	1.74	1.12
WT7 × 60	17.7	7.240	14.670	0.940	0.590	51.7	8.61	1.71	1.24	247	33.7	3.74
× 41	12.0	7.155	10.130	0.855	0.510	41.2	7.14	1.85	1.39	74.2	14.6	2.48
× 34	9.99	7.020	10.035	0.720	0.415	32.6	5.69	1.81	1.29	60.7	12.1	2.46
× 24	7.07	6.985	8.030	0.595	0.340	24.9	4.48	1.87	1.35	25.7	6.40	1.91
× 15	4.42	6.920	6.730	0.385	0.270	19.0	3.55	2.07	1.58	9.79	2.91	1.49
× 11	3.25	6.870	5.000	0.335	0.230	14.8	2.91	2.14	1.76	3.50	1.40	1.04
WT6 × 60	17.6	6.560	12.320	1.105	0.710	43.4	8.22	1.57	1.28	172	28.0	3.13
× 48	14.1	6.355	12.160	0.900	0.550	32.0	6.12	1.51	1.13	135	22.2	3.09
× 36	10.6	6.125	12.040	0.670	0.430	23.2	4.54	1.48	1.02	97.5	16.2	3.04
× 25	7.34	6.095	8.080	0.640	0.370	18.7	3.79	1.60	1.17	28.2	6.97	1.96
× 15	4.40	6.170	6.520	0.440	0.260	13.5	2.75	1.75	1.27	10.2	3.12	1.52
× 8	2.36	5.995	3.990	0.265	0.220	8.70	2.04	1.92	1.74	1.41	0.706	0.773
WT5 × 56	16.5	5.680	10.415	1.250	0.755	28.6	6.40	1.32	1.21	118	22.6	2.68
× 44	12.9	5.420	10.265	0.990	0.605	20.8	4.77	1.27	1.06	89.3	17.4	2.63
× 30	8.82	5.110	10.080	0.680	0.420	12.9	3.04	1.21	0.884	58.1	11.5	2.57
× 15	4.42	5.235	5.810	0.510	0.300	9.28	2.24	1.45	1.10	8.35	2.87	1.37
× 6	1.77	4.935	3.960	0.210	0.190	4.35	1.22	1.57	1.36	1.09	0.551	0.785
WT4 × 29	8.55	4.375	8.220	0.810	0.510	9.12	2.61	1.03	0.874	37.5	9.13	2.10
× 20	5.87	4.125	8.070	0.560	0.360	5.73	1.69	0.988	0.735	24.5	6.08	2.04
× 12	3.54	3.965	6.495	0.400	0.245	3.53	1.08	0.999	0.695	9.14	2.81	1.61
× 9	2.63	4.070	5.250	0.330	0.230	3.41	1.05	1.14	0.834	3.98	1.52	1.23
× 5	1.48	3.945	3.940	0.205	0.170	2.15	0.717	1.20	0.953	1.05	0.532	0.841
WT3 × 10	2.94	3.100	6.020	0.365	0.260	1.76	0.693	0.774	0.560	6.64	2.21	1.50
× 6	1.78	3.015	4.000	0.280	0.230	1.32	0.564	0.861	0.677	1.50	0.748	0.918
WT2 × 6.5	1.91	2.080	4.060	0.345	0.280	0.526	0.321	0.524	0.440	1.93	0.950	1.00

Courtesy of The American Institute of Steel Construction.
*WT is the symbol for a structural T-section (cut from a W-section), followed by the nominal depth in inches, and the weight in pounds per foot of length.

GEOMETRIC PROPERTIES OF ROLLED STEEL SHAPES



Structural Tees (SI Units)

Designation*	Area (mm^2)	Depth of Tee (mm)	Flange Width (mm)	Flange Thickness (mm)	Stem Thickness (mm)	Axis Z-Z I (10^6 mm^4)	Axis Z-Z Z (10^3 mm^3)	Axis Z-Z r (mm)	y_c (mm)	Axis Y-Y I (10^6 mm^4)	Axis Y-Y Z (10^3 mm^3)	Axis Y-Y r (mm)
WT457 × 171	21805	455.9	418.3	32.0	19.3	389	1098	133	102	196	936	94.7
× 119	15160	457.3	304.8	25.9	16.5	308	914	142	120	61.2	403	63.5
WT381 × 98	12515	384.9	267.8	25.4	15.6	175	613	118	99.1	40.8	305	57.2
× 80	10260	378.8	266.1	19.3	13.8	145	524	119	102	30.4	228	54.6
WT305 × 77	9870	305.6	323.9	19.1	12.7	78.7	328	89.2	65.8	54.1	333	73.9
× 70	8905	308.7	230.3	22.2	13.1	77.4	333	93.2	75.9	22.7	197	50.3
× 63	8000	306.1	229.1	19.6	11.9	69.1	300	93.2	75.4	19.6	172	49.5
× 46	5875	301.5	178.8	15.0	10.9	54.5	256	96.3	87.9	7.16	80.3	35.1
WT229 × 57	7225	231.3	280.3	17.3	10.8	29.9	161	64.5	45.7	31.7	226	66.3
× 45	5690	231.6	191.9	17.7	10.5	26.9	152	68.8	54.9	10.4	109	42.9
× 37	4730	228.5	190.4	14.5	9.0	22.3	128	68.6	53.8	8.32	87.7	41.9
× 30	3795	227.3	152.8	13.3	8.0	18.6	110	70.1	58.2	3.98	51.9	32.3
WT203 × 74	9485	215.5	264.8	25.0	14.9	32.0	187	57.9	44.7	38.8	293	63.8
× 37	4755	206.5	179.6	16.0	9.7	17.6	111	61.0	48.0	7.74	86.2	40.4
× 30	3800	203.3	177.7	12.8	7.7	13.8	87.7	60.2	46.0	5.99	67.5	39.9
× 19	2475	199.3	139.7	8.8	6.4	9.78	67.0	62.7	53.1	2.00	28.5	28.4
WT178 × 89	11420	183.9	372.6	23.9	15.0	21.5	141	43.4	31.5	103	552	95.0
× 61	7740	181.7	257.3	21.7	13.0	17.1	117	47.0	35.3	30.9	239	63.0
× 51	6445	178.3	254.9	18.3	10.5	13.6	93.2	46.0	32.8	25.3	198	62.5
× 36	4560	177.4	204.0	15.1	8.6	10.4	73.4	47.5	34.3	10.7	105	48.5
× 22	2850	175.8	170.9	9.8	6.9	7.91	58.2	52.6	40.1	4.07	47.7	37.8
× 16	2095	174.5	127.0	8.5	5.8	6.16	47.7	54.4	44.7	1.46	22.9	26.4
WT152 × 89	11355	166.6	312.9	28.1	18.0	18.1	135	39.9	32.5	71.6	459	79.5
× 71	9095	161.4	308.9	22.9	14.0	13.3	100	38.4	28.7	56.2	364	78.5
× 54	6840	155.6	305.8	17.0	10.9	9.66	74.4	37.6	25.9	40.6	265	77.2
× 37	4735	154.8	205.2	16.2	9.4	7.78	62.1	40.6	29.7	11.7	114	49.8
× 22	2840	156.7	165.6	11.2	6.6	5.62	45.1	44.5	32.3	4.25	51.1	38.6
× 12	1525	152.3	101.3	6.7	5.6	3.62	33.4	48.8	44.2	0.587	11.6	19.6
WT127 × 83	10645	144.3	264.5	31.8	19.2	11.9	105	33.5	30.7	49.1	370	68.1
× 65	8325	137.7	260.7	25.1	15.4	8.66	78.2	32.3	26.9	37.2	285	66.8
× 45	5690	129.8	256.0	17.3	10.7	5.37	49.8	30.7	22.5	24.2	188	65.3
× 22	2850	133.0	147.6	13.0	7.6	3.86	36.7	36.8	27.9	3.48	47.0	34.8
× 9	1140	125.3	100.6	5.3	4.8	1.81	20.0	39.9	34.5	0.454	9.03	19.9
WT102 × 43	5515	111.1	208.8	20.6	13.0	3.80	42.8	26.2	22.2	15.6	150	53.3
× 30	3785	104.8	205.0	14.2	9.1	2.39	27.7	25.1	18.7	10.2	99.6	51.8
× 18	2285	100.7	165.0	10.2	6.2	1.47	17.7	25.4	17.7	3.80	46.0	40.9
× 13	1695	103.4	133.4	8.4	5.8	1.42	17.2	29.0	21.2	1.66	24.9	31.2
× 7	955	100.2	100.1	5.2	4.3	0.895	11.7	30.5	24.2	0.437	8.72	21.4
WT76 × 15	1895	78.7	152.9	9.3	6.6	0.733	11.4	19.7	14.2	2.76	36.2	38.1
× 9	1150	76.6	101.6	7.1	5.8	0.549	9.24	21.9	17.2	0.624	12.3	23.3
WT51 × 10	1230	52.8	103.1	8.8	7.1	0.219	5.26	13.3	11.2	0.803	15.6	25.4

Courtesy of The American Institute of Steel Construction.

*WT is the symbol for a structural T-section (cut from a W-section), followed by the nominal depth in mm, and the mass in kg per meter of length.

APPENDIX D-1

LABORATORY REPORT ON TENSION TESTING

INTRODUCTION:

The purpose of this laboratory experiment was to investigate the behavior of a material under tensile loading. We investigated the failure of 6061-T6 aluminum subjected to tensile forces. A photograph of the tensile specimens employed is shown in Fig. LRT 1.

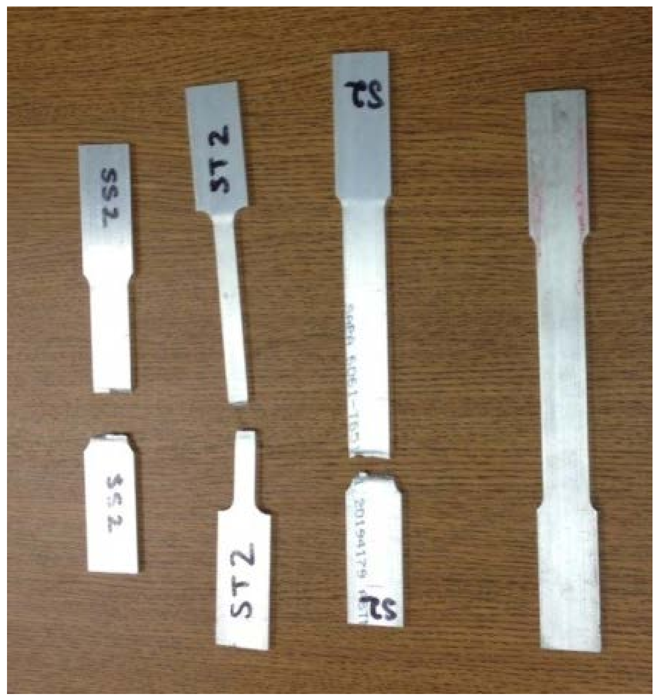


Fig. LRT 1 Aluminum 6061-T6 tensile specimens before and after failure.

The specimens shaped like "dog-bones" were machined from flat aluminum stock. Four specimens with dimensions given in the Table LRT 1 were tested to failure. The failure load is also presented in Table LRT 1.

Table LRT 1
Specimen Dimension and Failure load

Specimen	Thickness (in)	Width (in)	Cross sectional area (in^2)	Length (in)	Max Load (lbs)
S1	0.123	0.373	0.045879	2.249	2060
L1	0.128	0.381	0.048768	4.366	2078
L2	0.128	0.374	0.047872	4.51	2049
F1	0.25	0.379	0.09475	4.411	4262

Three of the four specimens were longer (by almost two times) than specimen identified as S1 in Table LRT 1. Most of the specimens were 0.125 in. thick by 0.375 in. wide. Specimens L1 and L2 had about the same cross section area at the narrowest part. They were tested to determine the difference in failure load for nearly identical specimens. Specimen S1 had the same cross sectional area as L1 and L2, but was about half as long. Specimen F1 was nearly the same length as L1 and L2, but was twice as thick (0.250 in. versus 0.125 in.).

PROCEDURE

The universal testing machine presented in Fig. LRT 2 was employed to apply an axial tensile load to the tension specimens. A load cell on the Instron testing machine recorded the load and the displacement of the cross head was also recorded. The cross head displacement was used to determine the elongation of the specimen.

Fig. LRT 2 The Instron machine used to apply an axial tensile force to the tensile specimens.

RESULTS AND DISCUSION

The load deflection curves for two nearly identical specimens — L1 and L2 are presented in Fig. LRT 3.

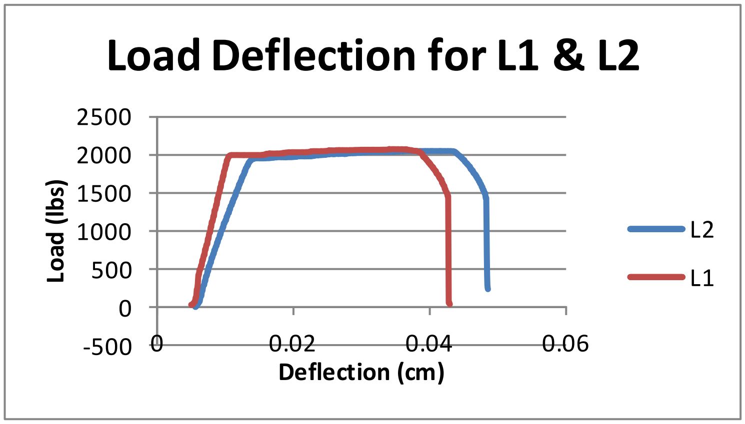


Fig. LRT 3 Load as a function of deflection for specimens L1 and L2.

The vertical axis in Fig. LRT 3 is the applied load in lbs as recorded by the load cell. The horizontal axis in this figure is the cross head displacement on the Instron machine measured in cm. Inspection of Fig. LRT 3 shows that the load deflection curves for the two specimens are nearly identical; however, slight differences are noted in the specimen elongation prior to failure. The difference can be attributed to some small slippage in the grips that are being used to hold the specimens and to the initial settings on the LabVIEW software that is being used to record the results. For all practical purposes, the two specimens behave in a similar manner, with nearly identical failure loads of 2049 and 2078 lbs.

The load deflection curves for the three specimens S1, L1 and F1 are presented in Fig. LRT 4.

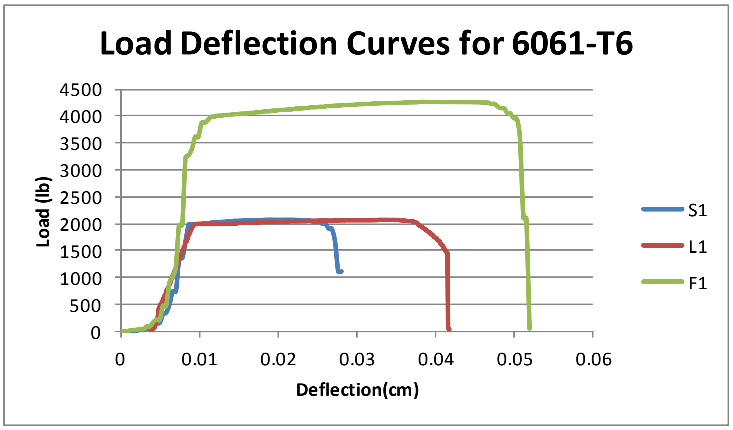


Fig. LRT 4 Load deflection curves for specimens S1, L1 and F1.

In Fig. LRT 4, S1 is the notation for the short specimen, L1 for the long specimen, and F1 for thick (fat) specimen. Although the material in all three specimens is the same, the load deflection curves are very different. The curves for each specimen, however, make sense. The specimen F1 is twice as thick as the other two specimens and supports about twice the load. Specimen S1 that is half as long as specimen L1 does not elongate as much.

Engineers seek methods for classifying materials to determine if they have sufficient strength to support the loads imposed on a structure. Engineers are also concerned with displacement and must determine if the structure undergoes excessive deflections under specified loads. The approach followed by engineers is to normalize the applied load with the cross sectional area. The applied force P is divided the specimen's cross sectional area A to give the normal stress as previously described in Chapter 5.

$$\sigma = P/A \qquad (1)$$

This normalization accounts for differences in the cross sectional area of the specimen. With regard to the specimen's displacement, engineers normalize it by dividing the increase in the specimen's length by its original length. This approach yields the normal strain as described in Chapter 5.

$$\varepsilon = \delta/L \qquad (2)$$

where ε is the normal strain and δ is the change in length of the specimen and L is its original length.

If we calculate the stress σ and the normal strain ε using the results from the experiments with specimens S1, L1 and F1, we are able to establish engineering stress – engineering strain curve as shown in Fig. LRT 5. In this figure the curves have been shifted to a common zero strain and zero stress.

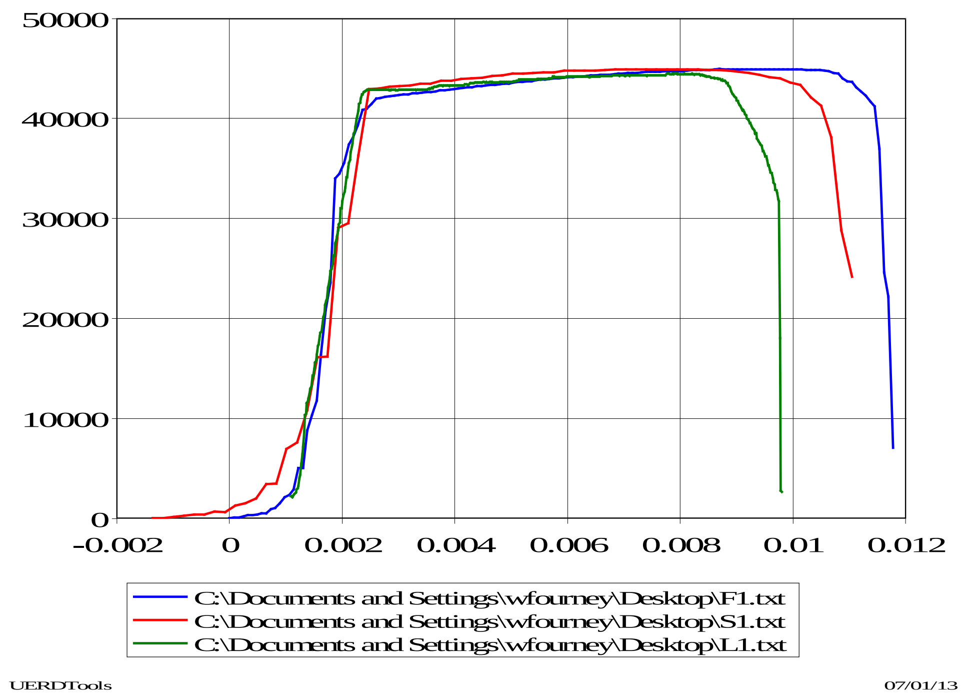


Fig. LRT 5 Engineering stress – engineering strain curves for specimens S1, L1 and F1.

The stress (in psi) is shown along the vertical axis and the strain in (in./in.) is shown along the horizontal axis. All of the curves should start at zero strain but some slight slippage of the grips results in the initial curvature of all three curves at low stresses. At higher stress levels the stress-strain curve is linear as expected from the description presented previously in Chapter 5.

The linear portion of the stress-strain curves represents the elastic response of the aluminum, and the slope of the curve gives its stiffness (modulus of Elasticity, E). The slope of specimen L1 is determined from the graph in Fig. LRT 6. We have only considered the linear region of the data and have fitted a straight line to the points plotted. A linear equation fitted the data with a correlation coefficient of 0.9961 (an excellent fit) is shown in the figure . The modulus of elasticity E was equal to 10×10^6 psi. From the stress strain curve in Fig. LRT5, the yield strength is approximately 42,000 psi. The ultimate stress is slightly greater at 45,000 psi. Observing the large strain at failure (about 10 to 12%), we can conclude that this aluminum exhibits good ductility. Note also that the area under the stress-strain curve is significant indicating that aluminum 6061-T6 can absorb large amounts of energy before failing in tension.

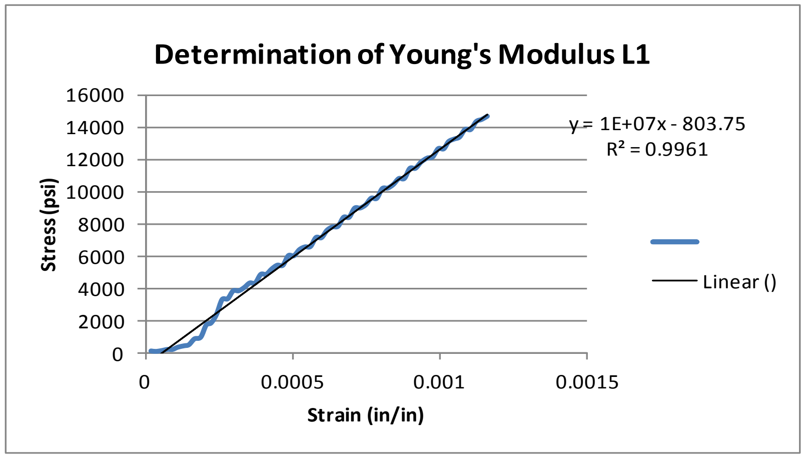


Fig. LRT 6 The slope of the linear region of the stress-strain curve gives the modulus of elasticity.

CONCLUSIONS

Four aluminum specimens with different dimensions were tested to failure by applying tensile forces. Three specimens had the same length but one of these had a larger thickness than the other two. The fourth specimen was shorter than the others. By testing two nearly identical samples, we showed that the scatter in our experimental data was small. The load deflection curves for the two nearly identical specimens were nearly the same. For the samples with different dimensions, we observed that the thicker specimen could carry about twice the load as the thinner specimens. For the shorter specimen, we found that it elongated less than the longer specimens. The three load deflection curves differed significantly, even though all were made from the same type of aluminum.

To be able to use load deflection data to design a structural component, engineers normalize the load by dividing it with the cross sectional area computing the normal stress $\sigma = P/A$. The deflection of the specimen is also normalized by dividing the change in length of the specimen by its original length to give the normal strain as $\varepsilon = \delta/L$. This normalization process enables us to construct an engineering stress-strain curve for the material. The stress-strain curve is unique for a given material. By examining the stress-strain curve we can determine its stiffness (modulus of elasticity). The modulus of elasticity enables us to calculate the amount of deformation (compared to other materials) when a stress is applied to a structural element. We can also determine the magnitude of the normal stress that can be imposed on a structural element before the material from which it is fabricate begins to plastically deform. It is also important to note the total area under the stress-strain curve, because this area provides a measure of the ductility of the material as compared to other materials. The larger the area, the more plastic deformation the structural element can undergo prior to failure. This area also indicates the energy that the material can absorb prior to failure.

APPENDIX D-2

LABORATORY REPORT ON BUCKLING

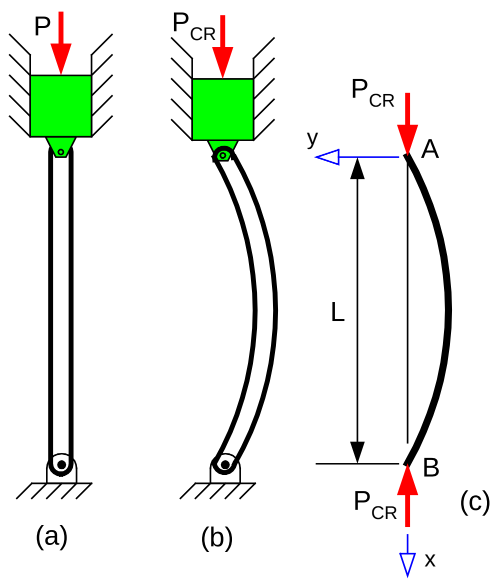
INTRODUCTION:

The purpose of this experiment was to investigate, experimentally, the Euler buckling equation as presented in the text. The Euler equation that is used to determine the critical buckling load for a column in compression with pinned ends is given below.

$$P_{CR} = \frac{\pi^2 EI}{L^2} \qquad (1)$$

A sketch of a column with both ends pinned is presented in Fig. LR1.

Fig. LR1 Axial loading of a column with both ends pinned.



$$P_{CR} = \frac{\pi^2 EI}{(kL)^2} \qquad (2)$$

The compressive load P_{CR} that will cause the column to snap out of its axial alignment is shown in Fig. LR1. For loads less than P_{CR}, the column essentially remains straight. When the load P_{CR} is reached the column suddenly deflects and any additional load causes it to collapse. A column pinned at both ends is free to rotate at the pin but cannot deflect to the left or right. With pinned ends there is no moment restraint on the ends of the column. For other end conditions, Eq. (2) is employed to account for the constraint at the column ends. The values for k are given for various end conditions in the table below.

Case No.	End Constraints	k
1	Pinned—Pinned	1
2	Pinned—Built-in	0.7
3	Free—Built-in	2
4	Built-in—Built-in	0.5

We will only investigate the pinned-pinned case in this experiment.

PROCEDURE

In the column buckling tests, we were trying to determine the relationship that three parameters have on the buckling load of a column pinned at both ends. These parameters included: the modulus of elasticity, the moment of inertia, and the column's length. The columns were fabricated from of three different materials — wood (oak), aluminum and steel. A universal testing machine shown in Fig. LR2 was employed to apply an axial compressive load to the columns. A load cell on the Instron testing machine recorded the critical buckling load at which each column failed. Wooden blocks with a V groove are attached to the machine at each end of the column prevent the specimens from moving sidewise as the load is applied.

Fig. LR2 The Instron Machine with wooden column ready for the test.

Three of the oak specimens are presented in Fig. LR3.

Fig. LR3. Column specimens fabricated from oak with longitudinal grain orientation.

After each test we compared the Euler buckling load (theoretical load) with the buckling load measured in the experiment to determine the accuracy of Eq. (1). Then we prepared a graph of the relationship between modulus of elasticity, moment of inertia, and length and the buckling load. Each graph showed one of the three parameters with the buckling load (i.e. the modulus of elasticity versus the buckling load, the moment of inertia versus the buckling load, and the length squared versus the buckling load).

Fig. LR4 Specimen ends were rounded or sharpened to simulate the pinned end condition.

For the series of tests the specimens fabricated from oak, some specimens were made with round ends and others had sharpened ends. We sought to determine if details of the ends would affect the results. We were attempting to provide pinned-pinned ends for each of the columns. The samples were of three different lengths and some were of different cross sectional areas but all had the same modulus of elasticity (1.8×10^6 psi for oak). The dimensions of the columns and the calculation of the critical buckling loads from Eq. (1) are presented in Table LR1.

Table LR1
Comparison of experimental and theoretical results from a buckling experiment with wooden columns

			cross secction					
Specimen designation	**End Condition**	**Experimental Buckling load (lbs)**	base (in)	height (in)	**moment of Inertia (I)**	**length (L)**	**L^2**	**Euler Buckling Load (P_{crit})**
LB-1	sharp	120	1.283	0.323	0.0036029064	23.6875	561.09765625	114.0740270167
LB-2	round	142	1.226	0.326	0.0035396639	23.8125	567.03515625	110.8981468005
LN-1	round	146	0.934	0.316	0.0024559916	23.6875	561.09765625	77.7607917564
LN-2	sharp	105	0.932	0.316	0.0024507325	23.6875	561.09765625	77.594280425
SN-1	round	280	0.957	0.325	0.002737668	18.1875	330.78515625	147.0303572595
SN-2	sharp	198	0.954	0.325	0.0027290859	18.1875	330.78515625	146.5694470486
SS-1	round	510	0.957	0.329	0.0028400003	12	144	350.3709929721
SS-2	sharp	480	0.955	0.329	0.0028340651	12	144	349.6387651916

We show the comparison between the experimentally determined loads and the calculated (theoretical) loads in Fig. LR5. The correlation coefficient of 0.9558 shows good agreement, but the calculated loads are lower than those measured experimentally. This is probably due to a moment constraint applied at the ends by the V groove in the wooden blocks holding the column in the Instron machine. The columns with round ends in general can support higher loads than the columns with the sharp ends. In the graph in Fig. LR5, R refers to columns with round ends and S for columns with sharp ends.

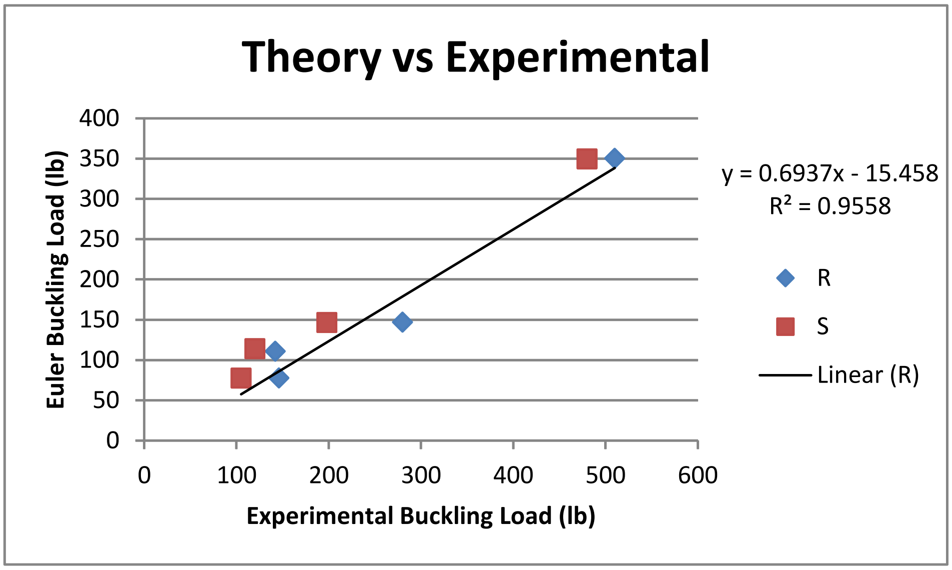


Fig. LR5 Comparison of theoretical and experimental results for column buckling with wooden columns.

Experiments were also conducted with three aluminum columns each having different lengths but with the same cross section. The ends on these specimens were flat. The modulus of elasticity for the aluminum was taken to be 10.4 × 10^6 psi. We repeated the procedure and calculated the Euler buckling load before testing each column in compression with the Instron machine. We repeated the tests on two of the columns (AL-1 and AL-2) to determine the scatter in the data. The measured buckling load and the Euler buckling load for each of the aluminum columns are given in Table LR2.

Table LR2
Comparison of experimental and theoretical results from a buckling experiment with Aluminum columns

Specimen Designation	Trial Number	End Condition	Experimental Buckling load (lbs)	width (in)	height (in)	Moment of Inertia,I (in^4)	Length, L (in)	L^2	Euler Buckling Load, P_{crit} (lbs)
AL-1	1	pin-pin	148	1	0.125	0.0001627604	10.9375	119.62890625	139.65130935
	2		146						
AL-2	1	pin-pin	85	1	0.125	0.0001627604	15.875	252.015625	66.2908634888
	2		78						
AL-3	1	pin-pin	34	1	0.125	0.0001627604	21.9375	481.25390625	34.7141772294

The results of the experiments with the aluminum columns are presented in Fig. LR6. Again there is a good agreement as reflected by a correlation coefficient of 0.9703. There is very little difference between the theoretical and the experimental results. Although there were differences in the results when two of the columns were tested twice, the difference was small. These results show that our data scatter was small and that buckling is an elastic phenomenon. After buckling if the load is removed, the column returns to its original shape. The aluminum in the buckled state remained in the elastic range and could be safely used again.

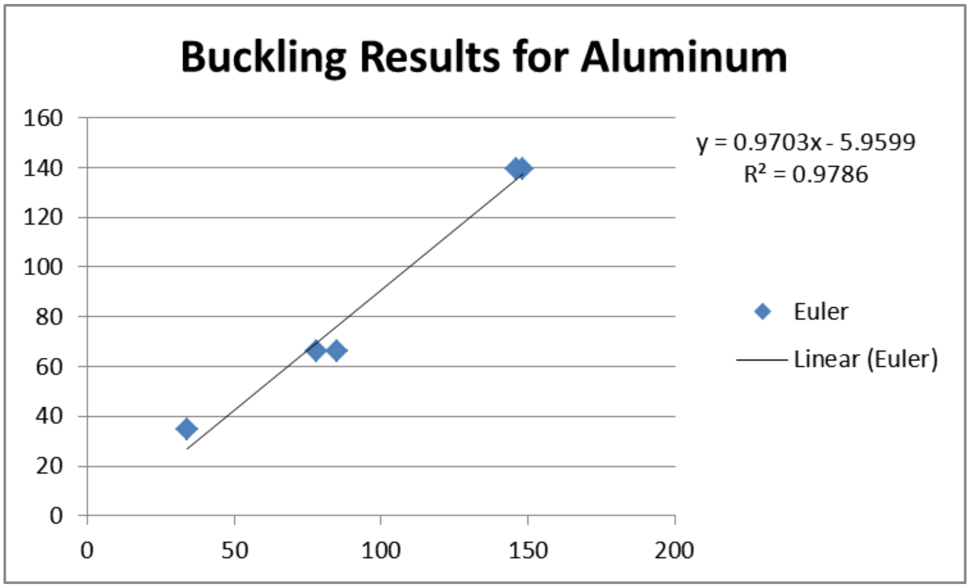


Fig. LR6 Comparison of theoretical and experimental results for column buckling with aluminum columns.

The third series of column buckling experiments involved testing 22 in. long oak columns and 22 in. long steel columns with the same cross-sectional area. The 22 in. length of these columns was the same as the length of the longest aluminum column discussed previously. This selection of column length enabled us to study the effect of modulus of elasticity on the buckling load. We used a modulus of elasticity for the steel columns of 30×10^6 psi. The experimental measurements and theoretical determination of the Euler buckling loads for each of the columns are given in Table LR3.

Table LR3
Comparison of experimental and theoretical results from a buckling experiments with Oak and Steel columns

Oak Samples:

Specimen designation	End Condition	Experimental Buckling load (lbs)	base (in)	height (in)	moment of Inertia (I)	length (L)	L^2	Euler Buckling Load (P_{crit})
LNN_1	sharp	13.7	1	0.125	0.0001627604	22	484	5.9737915543
LNN-2	sharp	13.7	1	0.125	0.0001627604	22	484	5.9737915543

Steel Samples:

Specimen Designation	Experimental Buckling load (lbs)	width (in)	height (in)	Moment of Inertia,I (in^4)	Length, L (in)	L^2	Euler Buckling Load, P_{crit} (lbs)
trial 1	96	1	0.125	0.0001627604	22	484	99.5688973732
trial 2	97	1	0.125	0.0001627604	22	484	99.5688973732

A graph of the experimentally determined buckling load versus the theoretically determined buckling load for the three 22 inch long specimens – wood, aluminum, and steel is presented in Fig. LR7. The experimental results correspond closely with the theoretical values, and the correlation coefficient of 0.9971 is excellent.

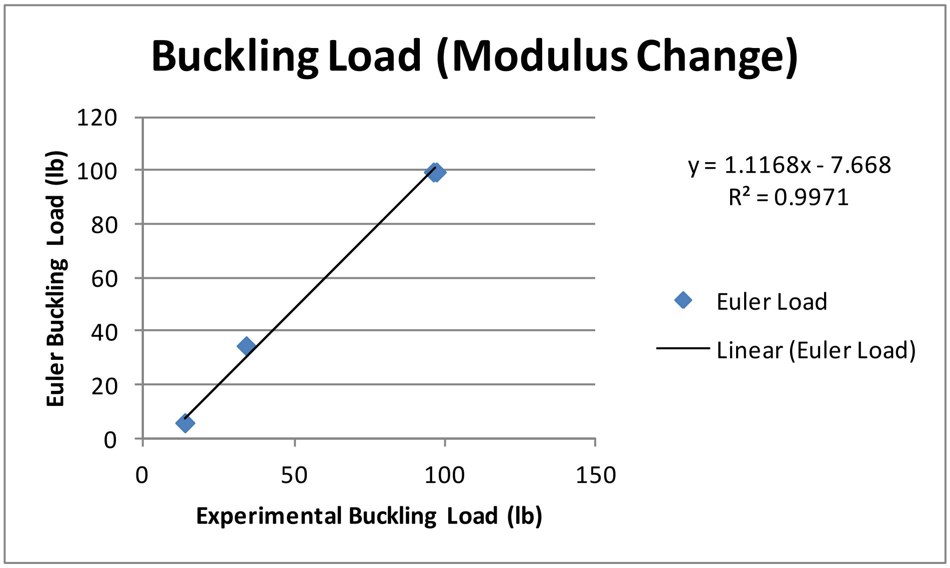


Fig. LR6 Comparison of theoretical and experimental results for column buckling with columns having different elastic modulus.

FURTHER ANALYSIS

There are two ways to assess the accuracy of the buckling experiments. The first is to compare the theoretical loads to the experimental loads to assess the validity of Eq. (1). We have made these comparison in tables LR1, LR2 and LR3. The second method of analysis is to prepare a graph of the experimental results for the buckling load against the square of the column length, the modulus of elasticity, and the minimum moment of inertia of the column cross section. These graphs yield a regression line, which indicates the correlation of the buckling load with the three parameters in the Euler equation (Eq. 1).

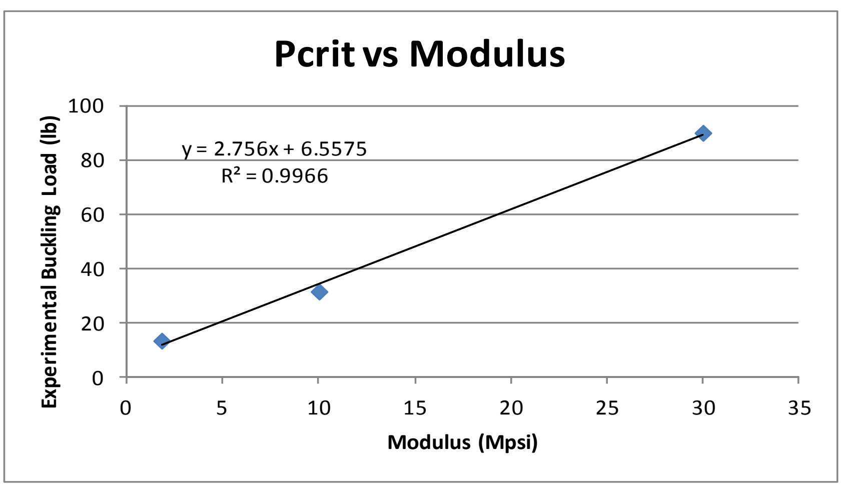


Fig. LR7 Experimentally determined column buckling load versus the elastic modulus of the columns.

The experimentally determined buckling load as a function of the modulus of elasticity is shown in Fig. LR7. These experiments were conducted using wood, aluminum and steel columns with the same cross section (0.125 in. by 1.0 in.) and the same length (22 in.). The correlation coefficient was 0.9966 indicating excellent correlation. We can state with high confidence that the critical buckling load is linearly related to elastic modulus E.

The experimentally determined buckling load for oak columns as a function of the moment of inertial is shown in Fig. LR8. The columns were of the same length. In this case the correlation coefficient is 0.8647, which is less than for previous results. However, the results indicate that buckling load is linearly related to the moment of inertia I. We believe the scatter in the data is due to the difference in the modulus of elasticity from column to column. Wood is not as homogeneous as metals such as steel or aluminum and differences in modulus and strength are to be expected. We noticed differences previously in the results for wood where the experimental determined buckling loads were higher than the theoretical values.

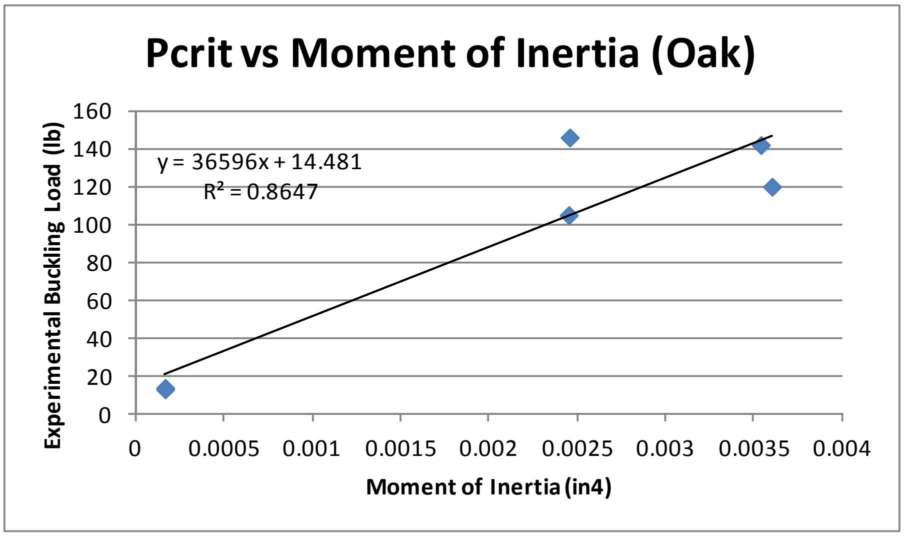


Fig. LR8 Experimentally determined column buckling load versus the moment of inertia of the columns.

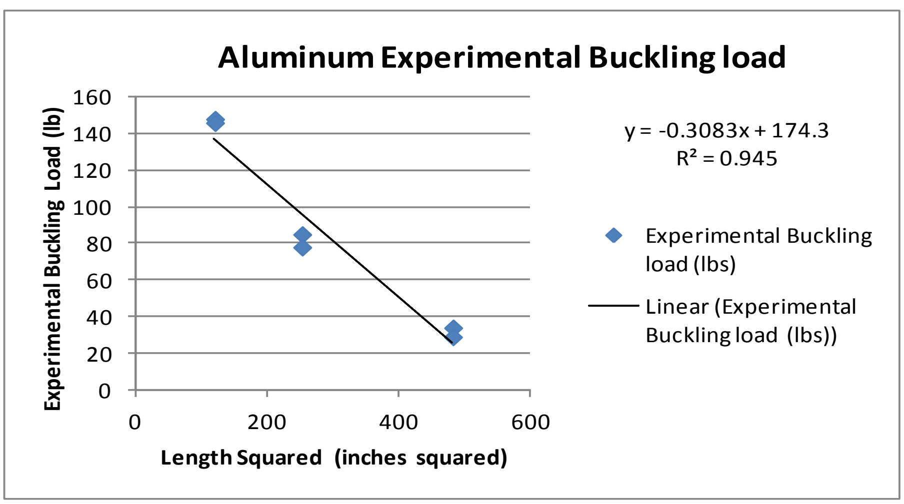


Fig. LR9 Experimentally determined column buckling load versus the square of their length.

Finally, the experimental buckling loads determined for the aluminum samples are shown as a function of the square of the length of the columns. Again, there is a good correlation with a correlation coefficient of 0.945. This result indicates that the buckling load is related to the length squared of the samples. In this case the buckling load decreases as the length increases indicating that the square of the length of the column must be in the denominator of the Euler equation.

CONCLUSIONS

The results of these experiments have shown that buckling is an elastic phenomena. The experiments demonstrate the concept of elastic instability. The experiments also show how buckling load relates to the modulus of elasticity, length, and moment of inertia of a column. We have observed the behavior of columns loading in compression until they buckle. We have established that the Euler buckling equation is reliable for predicting buckling loads for homogeneous materials. As is the case with other laboratory experiments we find that results obtained with wooden specimen do not correspond as closely to theoretical values calculated. However, using an adequate safety factor negates these uncertainties and ensures that the column undergoing a load would not buckle.

It is important that engineers run many tests, especially when they are using wood, which can be very unpredictable. It is essential that the design of the structure employ sturdy columns that will not buckle under the imposed loadings. Buckling is one of the more dangerous modes of failure. A structural element can undergo because under perfect conditions the structural element will not show signs of buckling until it passes the critical load and then it is too late.

APPENDIX E

FORCE AND MOMENT VECTORS

E.1 INTRODUCTION

Complex structures, vehicles and machines are constructed from many different structural elements such as rods, columns, struts, beams, cables, etc. To properly size these elements so they will perform their function safely for the duration of their design life, we must determine the forces and the moments acting at each point on each element. These forces and moments are vector quantities that are defined by specifying both their magnitude and direction. In this appendix, two different approaches for dealing with vector quantities are described:

1. A relatively simple approach based on trigonometry, which is easy to use when analyzing two-dimensional (plane) structures.
2. A more elegant technique utilizing vector algebra, which is useful in analyzing three-dimensional structures. This technique requires an understanding of a mathematical topic that may be new to many engineering students.

We will solve two-dimensional examples employing the trigonometric techniques. We will also introduce vector algebra and demonstrate the vector dot and cross products to show their application in determining unknown forces and moments in three dimensional structures.

E.2 INTERNAL AND EXTERNAL FORCES

In dealing with forces, we distinguish between those that are applied to the structure **(external)**, and those that develop within a structural element **(internal)**. The external forces include the active loads applied to the structure, such as those shown in Fig. E.1a-c, and the reaction forces, shown in Fig. E.1d, that develop at the supports maintaining the structure in equilibrium.

In Fig. E.1a, a simply supported beam is loaded with a **concentrated force** at a local point near its center. A concentrated force, applied at a point, is an idealization. Forces are always distributed over some area; however, with concentrated forces, we assume that the area is so small that it approaches a point. The symbol F is used to designate the magnitude of concentrated forces.

In Fig. E.1b, the beam is loaded with **uniformly distributed forces** that are applied over most of its length. Uniformly distributed forces along beams are specified in terms of force/unit length (i.e. lb/ft or N/m). The symbol q is used to designate the magnitude of the distributed forces applied to a beam.

Distributed forces that are increasing, as we move from the left end of the beam to its right, are illustrated in Fig. E.1c. Again, the symbol q is used to designate the magnitude of the distributed forces; however, in this case we must recognize that q is a function of position x along the length of the beam, designated by q(x).

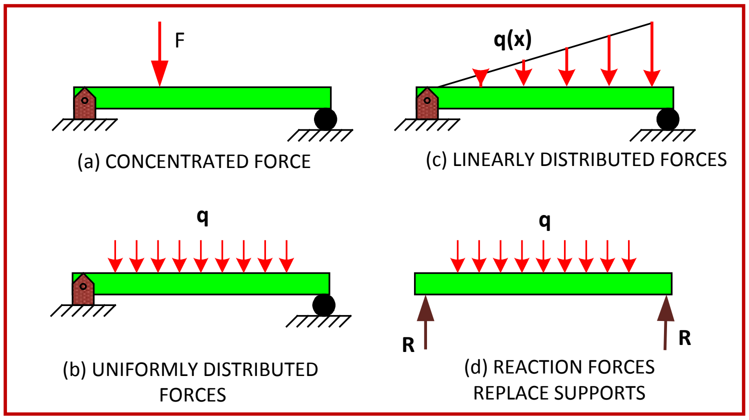


Fig. E.1 Examples of different types of external forces applied to the structure (beam).

The last example of external forces is shown in Fig. E.1d. The beam, with its uniformly distributed forces, is identical with that shown in Fig. E.1b, but its supports have been removed. The reaction forces developed by the supports to maintain the beam in equilibrium are shown as concentrated forces. The symbol R will be used to designate the magnitude of reaction forces.

Internal forces develop within a structural member due to the action of the applied external forces. These internal forces are not visible, although we try to visualize them by making imaginary cuts through a structural member. Let's examine a bar subjected to external forces F applied at each of its ends, as shown in Fig. E.2. We make a section cut perpendicular to the axis in the central region of the bar. This is an imaginary cut, not a real one, but it permits us to visualize either segment of the bar. We examine the segment on the left, and find the **normal stresses** σ, which are uniformly distributed over the area exposed by the section cut. When this stress is integrated over the area of the bar, an internal force P_{int} is generated that acts along the axis of the bar. The magnitude of P_{int} is given by:

$$P_{int} = \int \sigma \, dA \qquad \text{(E.1)}$$

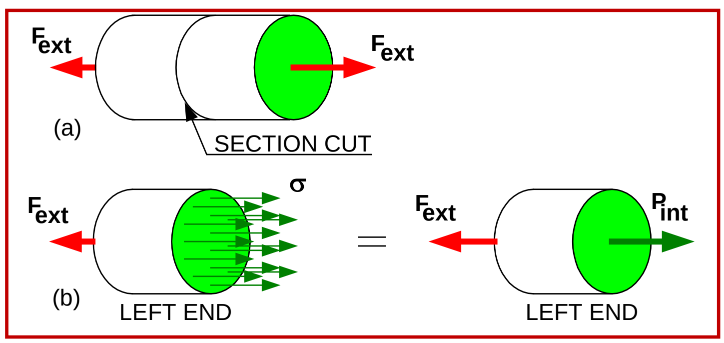


Fig. E.2 (a) A tension bar with external forces and a section cut dividing the bar into two parts.
(b) Two different representations of the left end of the bar.

The left end of the bar must be in equilibrium, which implies that:

$$\Sigma F_x = 0$$

Summing the forces in the x direction gives:

$$P_{int} = F_{ext} \qquad \text{(E.2)}$$

In this elementary example of a rod in tension, we have found the relation between the internal force P within the bar and the external forces F_{ext} applied at its ends. We will use the same three-step approach throughout this text in solving much more complex problems. In your solutions for internal forces to the assigned exercises, remember to:

1. Make an appropriate section cut.
2. Use the equations of equilibrium.
3. Solve for internal forces in structural members.

E.3 FORCE VECTORS

Forces are vector quantities, and as such it is necessary to specify both their magnitude and direction. Thus far, we have focused on their magnitude to illustrate the physical aspects of different types of forces. Let's now move to a more mathematical discussion of forces and vectors. A typical force vector is presented with an arrow as shown in Fig. E.3.

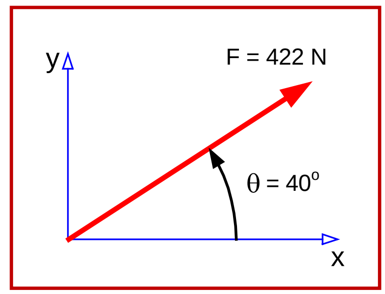


Fig. E.3 A force vector is represented by an arrow with a length proportional to its magnitude (422 N). The direction of the vector ($\theta = 40°$) is specified relative to the positive x-axis.

The length of the arrow indicates the magnitude, and the orientation of the arrow relative to the positive x-axis gives the direction. The tail end of the arrow is positioned at the origin of the (x-y) coordinate system, with the arrowhead at its tip.

EXAMPLE E.1

Construct a graph showing a force vector with a magnitude of 4,100 N and a direction of 150°, relative to the positive x-axis, in an x-y coordinate system.

Solution:

First, define an x-y coordinate system making certain to include the negative side of the x-axis. Next, layout a line from the origin inclined by 180° – 150° = 30° relative to the negative x-axis. Select a scale for the magnitude; we suggest 1 mm = 100 N. Measure a length of 41 mm along this line, and place an arrowhead at this point. Place arrowheads on the x and y axes of the x-y coordinate system, and include the measure of the angle and the magnitude of the force vector. The result obtained is illustrated in Fig. EXPL.1:

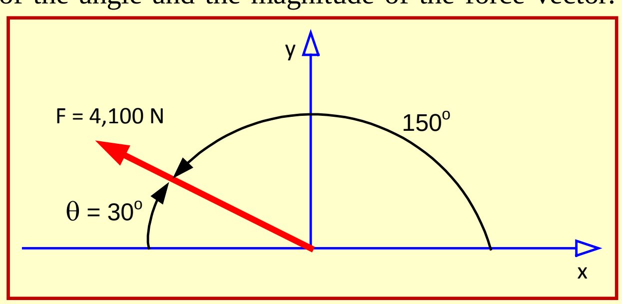
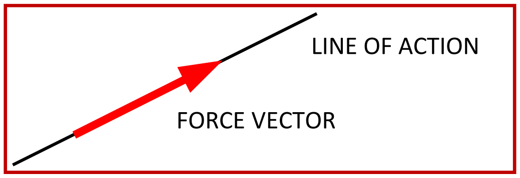


Fig. EXPL.1

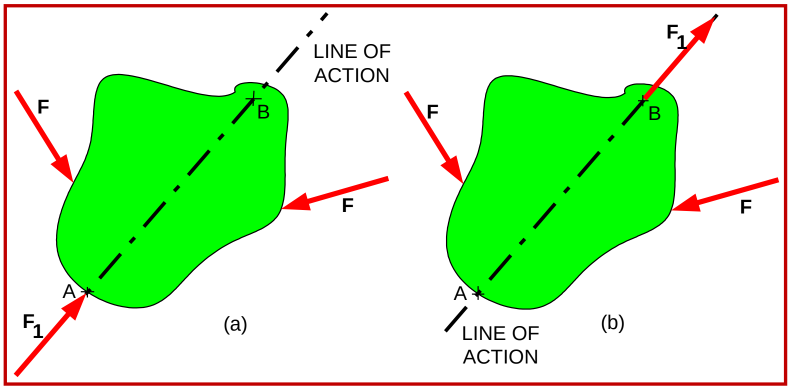
A force vector has a line of action, as indicated in Fig. E.4, which is collinear with the direction of the vector.

Fig. E.4 The line of action is collinear with the force vector.



It is permissible to move a force vector along its line of action, if it is convenient to do so. Shifting the force along the line of action does not change the state of equilibrium of the body. To illustrate this concept of the **transmissibility** of forces, suppose the body, shown in Fig. E.5a, is in equilibrium with the force F_1 applied at point A. The body remains in equilibrium as the force F_1 is moved along its line of action from point A to point B, as indicated in Fig. E.5b. In the first instance, we are pushing on the body with F_1, and in the second, we are pulling on the body. In both cases the force F_1 acts on the body with the same magnitude and direction; hence, the body remains in equilibrium.

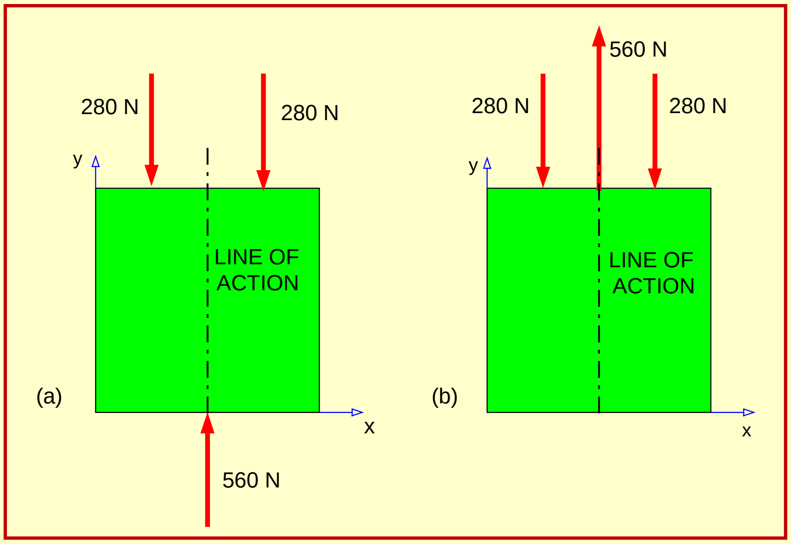
Fig. E.5 The body remains in equilibrium as $\mathbf{F_1}$ is shifted along the line of action from A to B.



EXAMPLE E.2

Let's consider two identical blocks with the applied forces, as shown in Fig. EXPL.2. Are both of the blocks in equilibrium after we shift the 560 N force along the line of action from the bottom edge of the block to its top edge?

Fig. EXPL.2



Solution:

The block in Fig. EXPL.2a is in equilibrium with a 560 N force applied to the bottom edge. From Eq. (1.2b), we write the equation for equilibrium in the y direction to obtain:

$$\Sigma F_y = 0$$

$$560 - 280 - 280 = 0$$

We can write the same equations for the block shown in Fig. EXPL.2b. The fact that the 560 N force has been shifted along the line of action does not change the equilibrium relation.

We have already made a distinction between applied (active) forces F and reactive forces R. To maintain the body in equilibrium, reactive forces develop at structural supports and increase in magnitude as the active forces are applied. The reactive forces are shown in free body diagrams (FBDs), when the supports are removed from a structural member. A typical FBD is illustrated in Fig. E.6, showing three reactive forces that maintain a uniformly loaded, simply supported beam in equilibrium. In constructing the FBD, we remove the supports and apply the reactive forces at their location. The magnitude of the reactive forces is determined from the equilibrium relations, which were discussed previously in Chapter 2. The directions of both R_{yL} and R_{yR} are perpendicular to the surface of the beam. The direction of R_{xL} is taken in the positive x direction. The two reactive forces R_{yL} and R_{xL} at the left support replace the effect of the pinned joint. It is emphasized that you must specify directions for the reactive forces that are consistent with the type of supports removed in constructing the FBD.

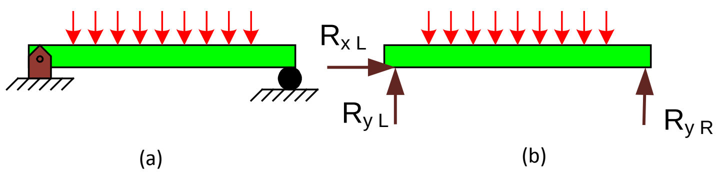


Fig. E.6 Reactive forces in (b) replace the supports when drawing a FBD of the beam in (a).

EXAMPLE E.3

To illustrate the direction of reactive forces, let's consider a circular pipe supported in a right-angled rack, as shown in Fig. EXPL.3. Draw a FBD of the pipe and show the angle that the reaction forces make with a horizontal line.

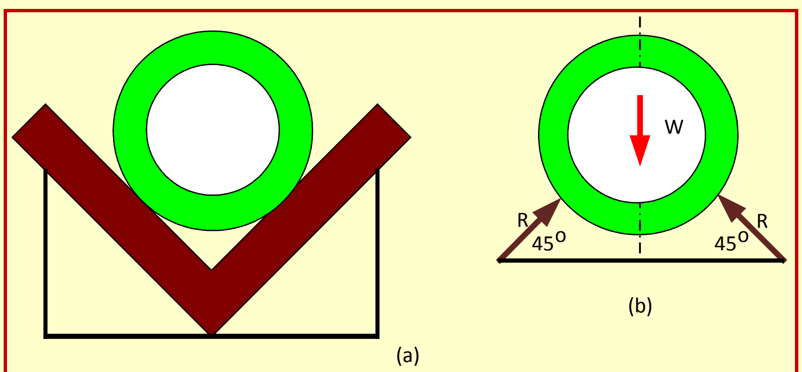


Fig. EXPL.3

Solution:

Remove the rack and support the pipe with a pair of reaction forces R. These forces are perpendicular to the surface of the pipe at the contact points, as shown in Fig. EXPL.3b. The lines of action of the reaction forces are radial and pass through the center of the pipe[1]. Because the rack is symmetric with a 90° included angle, it is easy to visualize the reaction forces at a 45° angle relative to a horizontal line. The active force W represents the weight of the pipe, which acts downward along the pipe's centerline.

E.4 ADDING AND SUBTRACTING VECTORS

Suppose that we have two force vectors **A** and **B**, shown in Fig. E.7a, that are to be added to give the vector sum $\mathbf{S_v} = \mathbf{A} + \mathbf{B}$. There are several approaches that we can follow to determine this vector sum; however, to begin let's visualize the process of adding vectors by examining Fig. E.7b. In this illustration, we shift vector **B** to a new position, while retaining its direction and magnitude. Its new position places the tail of vector **B** at the tip of vector **A**. We then construct a line from the tail of vector **A** to the tip of vector **B** to obtain both the magnitude and direction of the vector sum $\mathbf{S_v}$.

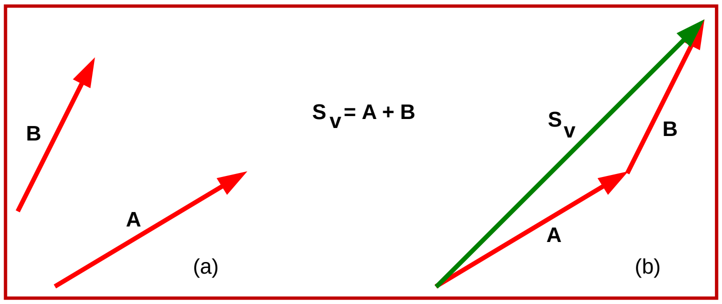


Fig. E.7 An illustration of the physical interpretation of vector addition $\mathbf{S_v} = \mathbf{A} + \mathbf{B}$.

Now that you can visualize adding **A** + **B**, let's add **B** + **A**. Will reversing the order of adding vectors **A** and **B** give the same result? Let's repeat the process of moving vectors to obtain the results shown in Fig. E.8. In this case, we shift vector **A**, maintaining its magnitude and direction, so that its tail coincides with the tip of vector **B**. The line from the tail of **B** to the tip of **A** is the vector $\mathbf{S_v}$. From these illustrations, it is evident that the order with which the vectors are added does not affect the result and:

$$\mathbf{S_v} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{(E.3)}$$

This relation establishes the **commutative property of vector addition**.

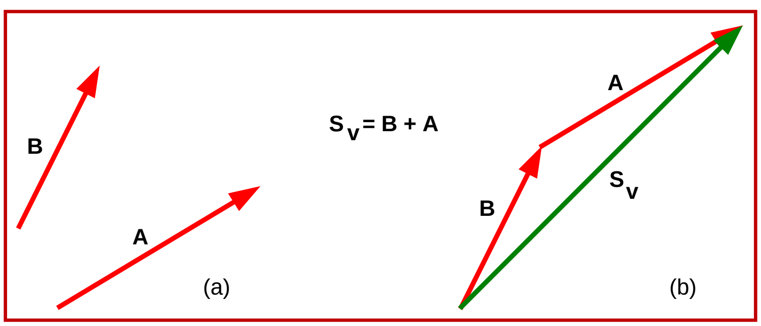


Fig. E.8 Illustration of the physical interpretation of vector addition $\mathbf{S_v} = \mathbf{B} + \mathbf{A}$.

[1] Friction forces at the contact points are assumed to be negligible.

With a graphical approach, it is possible to add vectors and estimate the magnitude and direction of $\mathbf{S}_v$ using only a scale and a protractor. We will not pursue the graphical technique in this text, because there is a more accurate trigonometric method discussed later.

Let's consider vector subtraction where we seek $\mathbf{D}_v$: the vector formed by **A – B**. The subtraction process is illustrated in Fig. E.9. In Fig. E.9a, the same two vectors **A** and **B** are shown. In Fig. E.9b, we rotate **B** though 180° to reverse its direction and obtain a new vector **– B**. Next, we add **– B** to **A** following the same procedure used for vector addition to determine $\mathbf{D}_v$ as shown in Fig. E.9c.

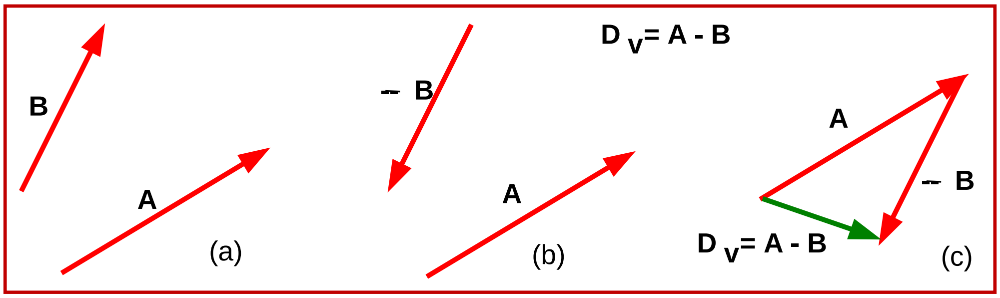


Fig. E.9 An illustration of the process of vector subtraction.

From Figs. E.7, E.8 and E.9, which illustrate the addition or subtraction of two vectors, it is clear that both $\mathbf{S}_v$ and $\mathbf{D}_v$ are sides of a triangle. The trigonometric properties of a triangle will be useful in determining the unknown magnitudes and directions of either $\mathbf{S}_v$ or $\mathbf{D}_v$ in vector addition or subtraction. The sides and the included angles of a triangle are defined in Fig. E.10.

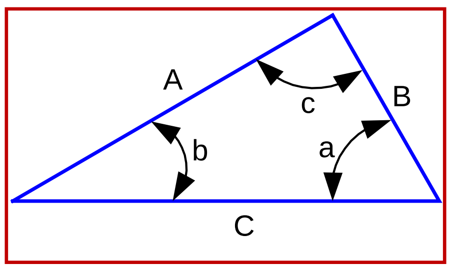
In analyzing triangles, we make use of the **sine law**:

$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)} \tag{E.4}$$

And the **cosine law**:

$$C = \sqrt{A^2 + B^2 - 2AB\cos(c)} \tag{E.5}$$

Fig. E.10 Triangle with sides A, B, and C and included angles of a, b, and c.

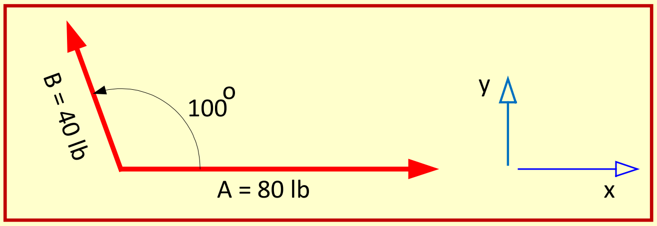


The following examples illustrate the method for using the sine and cosine laws in adding two vectors.

EXAMPLE E.4

Add the vectors **A** and **B,** defined in Fig. EXPL.4, to obtain the vector sum $\mathbf{S_v}$.

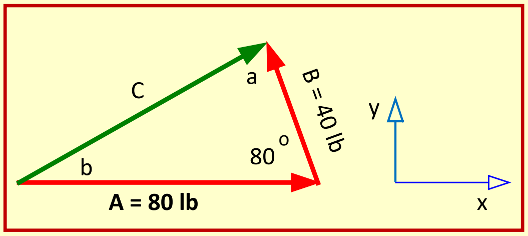
Fig. EXPL.4



Solution:

We begin by constructing an A, B and C triangle where the magnitude of $\mathbf{S_v}$ is the same as the side C. The construction of the triangle is presented in Fig. EXPL.4a.

Fig. EXPL.4a



We solve for the magnitude of C using the cosine law, which is given in Eq. (E.5).

$$C = \left[A^2 + B^2 - 2AB\cos(c)\right]^{1/2}$$

$$C = \left[(80)^2 + (40)^2 - (2)(80)(40)\cos(80°)\right]^{1/2} \qquad \text{(a)}$$

$$C = \left[8{,}000 - (6{,}400)(0.1736)\right]^{1/2} = \sqrt{6{,}889} = 83.0 \text{ lb}$$

Now that we have the magnitude of $\mathbf{S_v}$, the sine law given in Eq. (E.4) may be used to determine its direction. The direction of $\mathbf{S_v}$ relative to the x-axis is given by the angle b.

$$\frac{B}{\sin(b)} = \frac{C}{\sin(c)}$$

$$\sin(b) = \frac{B\sin(c)}{C} = \frac{(40)\sin(80°)}{83.0} = 0.4746 \qquad \text{(b)}$$

$$b = 28.33°$$

The direction of $\mathbf{S_v}$ is at angle of 28.33° relative to the x-axis. The solution is complete, because we have determined both the magnitude and the direction of the vector sum $\mathbf{S_v}$. Note, the results are given with four significant figures, and the units for both the magnitude and the direction are stated.

EXAMPLE E.5

Two forces **A** and **B** are applied to lift the weight shown in Fig. EXPL.5. If the weight W is to move vertically upward without swinging when it clears the floor, determine the angle θ to specify for the direction of the force **B**.

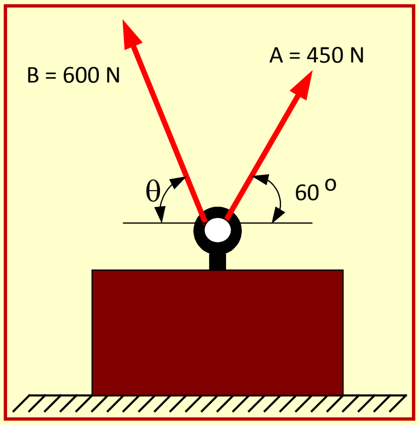


Fig. EXPL.5

Solution:

To begin the solution of this problem, draw the force triangle as shown in Fig. EXPL.5a. It is evident that the direction of $\mathbf{S_v}$, the resultant of the vector addition **A** + **B**, must be vertical if the weight is to be lifted without swinging.

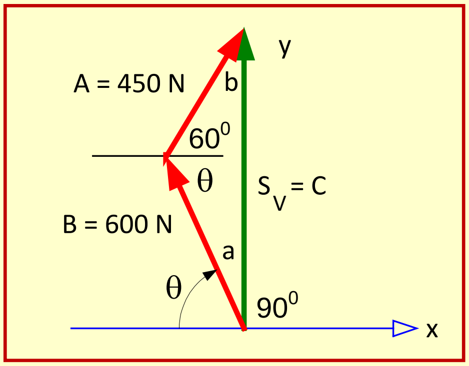


Fig EXPL.5a

Let's use the force triangle shown in Fig. EXPL.5a to solve for the angle θ. First, recall from your studies of plane geometry that the sum of the included angles within a triangle is 180°:

$$a + b + c = 180°$$

$$(90° – θ) + b + (θ + 60°) = 180°$$

$$b = 30° \qquad \text{(a)}$$

Next use the law of sines given by Eq. (E.4) to solve for θ.

$$\frac{A}{\sin a} = \frac{B}{\sin b}$$

$$\frac{450}{\sin(90° - \theta)} = \frac{600}{\sin(30°)} \qquad \text{(b)}$$

$$\sin(90° - \theta) = (3/4)(1/2) = 0.375$$

$$90° - \theta = 22.02° \Rightarrow \theta = 67.98°$$

We must apply the force B at an angle θ = 67.98° if the weight is to lift vertically from the floor without swinging.

EXAMPLE E.5 (Continued)

What weight can be lifted from the floor with the forces **A** and **B** applied as shown in Fig. EXPL.5?

Solution:

The magnitude of the vector sum S_v or the side C of our force triangle gives the weight that can be lifted. Let's use the law of cosines given by Eq. (E.5) to determine C.

$$C^2 = A^2 + B^2 - 2AB\cos(c)$$

$$C^2 = (450)^2 + (600)^2 - 2(450)(600)\cos(127.98°) \qquad \text{(a)}$$

$$C^2 = 8.948 \times 10^5 \Rightarrow C = 945.9\text{ N}$$

These results indicate that only 945.9 N can be lifted even though forces of 450 N and 600 N (a total of 1,050 N) were applied in the lifting process. Only the vertical components of the forces A and B contributed to lifting. The horizontal components cancelled each other, and did not contribute to lifting the weight. We will explore the components of a force in the next section.

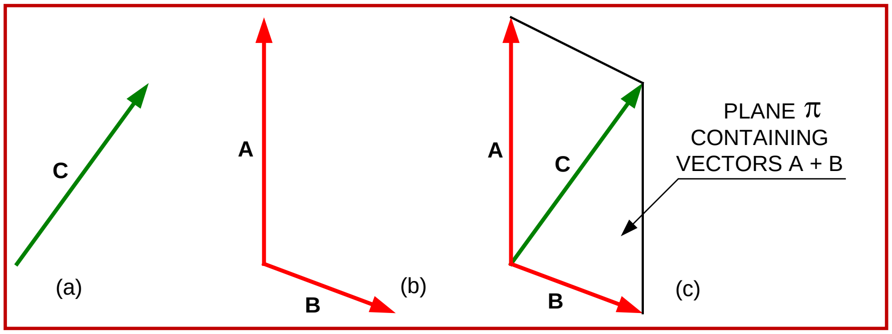
E.5 COMPONENTS OF A FORCE VECTOR

In Section E.4 the techniques for combining two force vectors into a vector sum $\mathbf{S}_v$ were discussed. Let's now reverse the process, and consider a technique for converting a single force vector into two equivalent vectors. Consider a force vector **C,** as shown in Fig. E.11a. This vector can be resolved into a set of two vectors **A** and **B** that is equivalent to **C**. Such a set of vectors is shown in Fig. E.11b. Clearly, an infinite set of selections of **A** and **B** can be made that are equivalent to **C**. The only requirement on the choice of **A** and **B** is that they satisfy the equation for vector addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \qquad \text{(E.6)}$$

We illustrate the concept of vector addition when resolving a single vector into two equivalent vectors with the parallelogram presented in Fig. E.11c. Note that the vector **C** and its equivalents **A** and **B** are coplanar — they all lie in the same plane π.

Fig. E.11 Resolution of a vector **C** into an equivalent vector pair **A** and **B**.



Next consider the special case of resolving a vector into an equivalent vector pair oriented along the Cartesian axes. Consider a force vector **F** that lies in the x-y plane, as illustrated in Fig. E.12a.

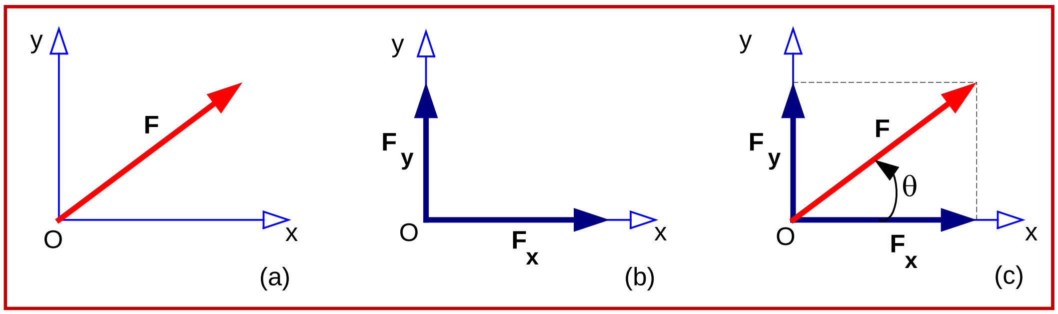


Fig. E.12 Resolution of vector **F,** lying in the x-y plane, into Cartesian force vectors $\mathbf{F_x}$ and $\mathbf{F_y}$.

The equivalent Cartesian force vectors $\mathbf{F_x}$ and $\mathbf{F_y}$ are shown in Fig. E.12b where

$$\mathbf{F} = \mathbf{F_x} + \mathbf{F_y} \qquad \text{(E.7)}$$

The magnitudes of the Cartesian force vectors $\mathbf{F_x}$ and $\mathbf{F_y}$ are given by:

$$F_x = F \cos\theta \qquad \text{and} \qquad F_y = F \sin\theta \qquad \text{(E.8)}$$

The relation between F and the Cartesian components F_x and F_y is given by:

$$F = [F_x^2 + F_y^2]^{1/2} \qquad \text{(E.9)}$$

Resolution of a vector into its Cartesian components is extremely important, as this concept will be used repeatedly in applying the equations of equilibrium. Next consider the following two examples to demonstrate the procedures used in determining the Cartesian components of a vector.

EXAMPLE E.6

Determine the x and y components of the 1,200 lb force shown in the Fig. EXPL.6. In addition check the solutions for F_x and F_y by verifying the value of 1,200 lb for the magnitude of the force F.

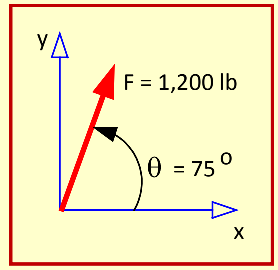


Fig. EXPL.6

Solution:

From Eqs. (E.8), we write:

$$F_x = F \cos \theta = (1{,}200) \cos (75°) = (1{,}200)(0.2588) = 310.6 \text{ lb}$$
$$F_y = F \sin \theta = (1{,}200) \sin (75°) = (1{,}200)(0.9659) = 1{,}159 \text{ lb} \qquad \text{(a)}$$

We use Eq. (E.9) to verify the value of 1,200 lb for the magnitude F.

$$F = [F_x^2 + F_y^2]^{1/2} = [(310.6)^2 + (1{,}159)^2]^{1/2} = 1{,}200 \text{ lb} \qquad \text{(b)}$$

EXAMPLE E.7

Determine the x and y components of the 55 kN force shown in Fig. EXPL.7. Also check the solutions for F_x and F_y by verifying the value of 55 kN assigned to F.

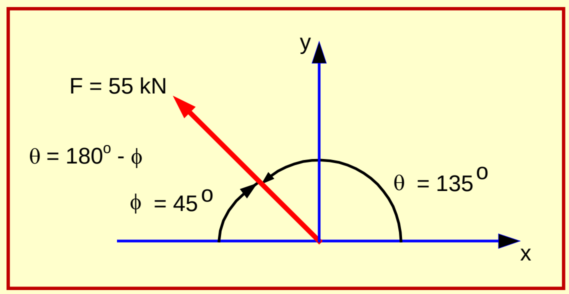


Fig. EXPL.7

Solution:

From Eqs. (E.8), we write:

$$F_x = F \cos \theta = (55) \cos (135°) = (55)(-0.7071) = -38.89 \text{ kN}$$
$$F_y = F \sin \theta = (55) \sin (135°) = (55)(0.7071) = +38.89 \text{ kN} \qquad \text{(a)}$$

We use Eq. (E.9) to verify the value of 55 kN for the magnitude F.

$$F = [F_x^2 + F_y^2]^{1/2} = [(-38.89)^2 + (38.89)^2]^{1/2} = 55.00 \text{ kN} \qquad \text{(b)}$$

E.5.1 Adding Three or More Coplanar Vectors

In the previous discussion, we considered the addition and subtraction of two vectors, **A** and **B**. A graphical method for performing the addition and subtraction was described and an analytical approach was discussed and demonstrated with examples. The analytical approach involved forming a triangle with sides A, B and C and employing the sine and cosine laws to solve for the unknown vector **C**. This approach is effective if you are required to add two vectors **A** and **B**, but the approach becomes tedious if it is necessary to add three or more vectors together. A much more efficient approach involves the use of adding the Cartesian components of each vector to form a vector sum.

Suppose we have the three vectors, **A**, **B** and **C** that lie in a common plane (coplanar), as shown in Fig. E.13. Let's form the vector sum $\mathbf{S_v} = \mathbf{A} + \mathbf{B} + \mathbf{C}$.

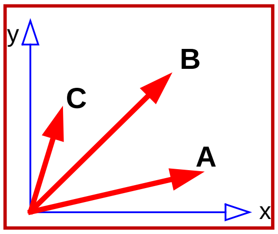


Fig. E.13 Three vectors A, B and C.

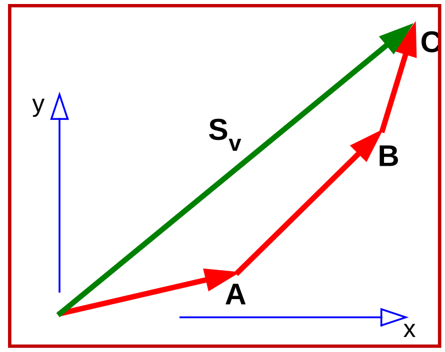


Fig. E.14 Vector addition of three vectors A, B and C by a graphical technique.

We may add the three vectors using a graphical technique by connecting them together tip to tail, as shown in Fig. E.14.

Suppose we attempt to perform the addition of the three vectors by using the triangle approach described previously in Section E.4. Clearly, the geometric shape in Fig. E.14 is not a triangle because it has four sides (tetragon). To employ the laws of sines and cosines, it is necessary to form two triangles by dividing the tetragon as illustrated in Fig. E.15.

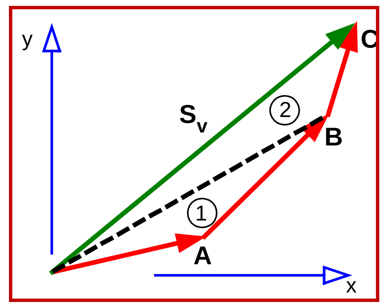


Fig. E.15 Dividing the vector tetragon into two triangles.

It is possible to solve for the vector sum $\mathbf{S_v}$ if we employ the sine and cosine laws twice; however, this approach is time consuming and mathematical errors are common. It is recommended that you resolve the three (or more) vectors into their Cartesian components, and add them in the manner described in the following paragraphs and examples.

First, resolve the vectors **A**, **B** and **C** and determine their Cartesian components, by using Eq. (E.8). These Cartesian components are illustrated in Fig. E.16.

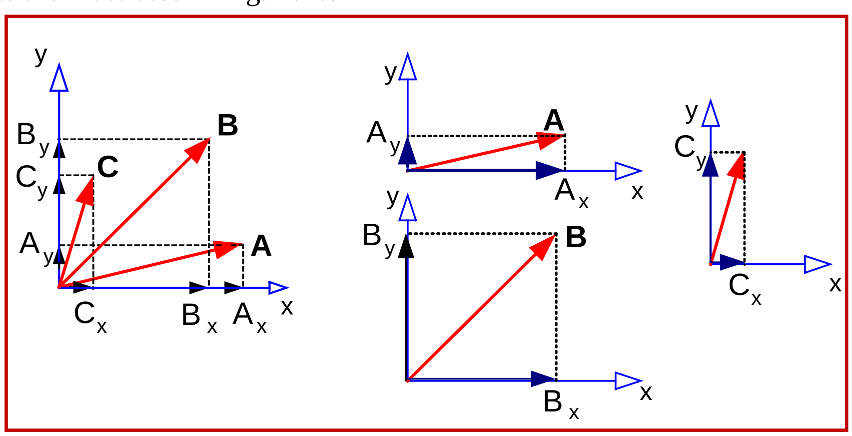


Fig. E.16 Resolving vectors **A**, **B**, and **C** into Cartesian components A_x, A_y, B_x, B_y, C_x and C_y.

Next add the Cartesian components together to obtain:

$$S_{vx} = A_x + B_x + C_x \quad \text{and} \quad S_{vy} = A_y + B_y + C_y \qquad \text{(E.10)}$$

Substituting Eq. (E.10) into Eq. (E.9) yields the magnitude of the sum of the three vectors as:

$$S_v = [(S_{vx})^2 + (S_{vy})^2]^{1/2} \tag{E.11}$$

The vector $\mathbf{S_v}$ and its Cartesian components S_{vx} and S_{vy} are presented in Fig. E.17. Also shown is the angle θ that defines the direction of the vector relative to the positive x-axis.

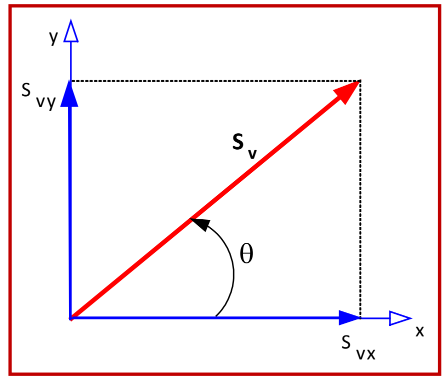


Fig. E.17 The vector $\mathbf{S_v}$ and its Cartesian components S_{vx} and S_{vy}.

The direction of the vector $\mathbf{S_v}$, given by θ, is determined from:

$$\theta = \sin^{-1}(S_{vy}/S_v) = \cos^{-1}(S_{vx}/S_v) = \tan^{-1}(S_{vy}/S_{vx}) \tag{E.12}$$

EXAMPLE E.8

Three vectors $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$ are illustrated in Fig. EXPL.8. Determine the sum of these three vectors and determine the angle θ the vector $\mathbf{S_v}$ makes with the positive x-axis.

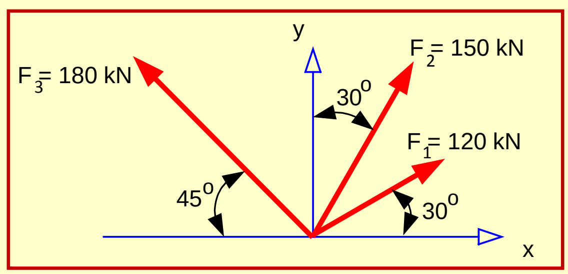


Fig. EXPL.8

Solution:

Determine the Cartesian components of the three forces using Eq. (E.8) as:

F_{1x} = (120) cos (30°) = 103.9 kN $\quad$ F_{1y} = (120) sin (30°) = 60.00 kN

F_{2x} = (150) cos (90° – 30°) = 75.00 kN $\quad$ F_{2y} = (150) sin (60°) = 129.9 kN $\quad$ (a)

F_{3x} = (180) cos (180° – 45°) = – 127.3 kN $\quad$ F_{3y} = (180) sin (135°) = 127.3 kN

Using Eq. (E.10) determine the Cartesian components of the vector sum $\mathbf{S_v}$ as:

$$S_{vx} = F_{1x} + F_{2x} + F_{3x} = 103.9 + 75.00 - 127.3 = 51.60 \text{ kN}$$

$$S_{vy} = F_{1y} + F_{2y} + F_{3y} = 60.00 + 129.9 + 127.3 = 317.2 \text{ kN} \tag{b}$$

The magnitude S_v is determined from Eq. (E.11) as:

$$S_v = [S_{vx}^2 + S_{vy}^2]^{1/2} = [(51.6)^2 + (317.2)^2]^{1/2} = (10.33 \times 10^4)^{1/2} = 321.4 \text{ kN} \qquad (c)$$

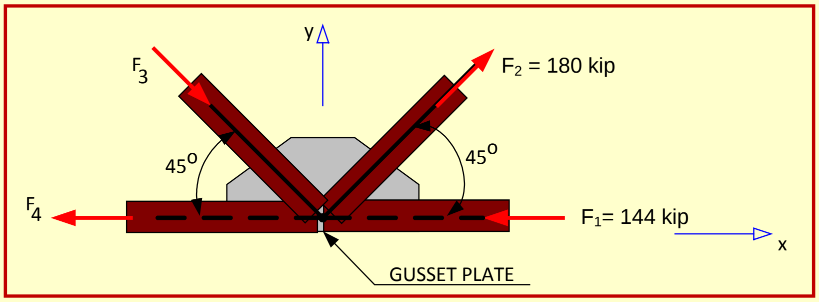
Finally, the direction of the vector $\mathbf{S_v}$ relative to the positive x-axis is given by Eq. (E.12) as:

$$\theta = \sin^{-1} (S_{vy} / S_v) = \sin^{-1} (317.2/321.4) = 80.73° \qquad (d)$$

EXAMPLE E.9

The gusset plate illustrated in Fig. EXPL.9 is subjected to four forces from the attached uniaxial structural members. If the vector sum of these four forces is zero, determine the forces F_3 and F_4.

Fig. EXPL.9



Solution:

The fact that the vector sum of the four forces $\mathbf{S_v} = 0$, implies that:

$$S_{vx} = S_{vy} = 0 \qquad (a)$$

Use Eq. (E.8), Eq. (E.10) and Eq. (a) to obtain:

$$S_{vy} = F_1 \sin (0°) + F_2 \sin (45°) - F_3 \sin (45°) + F_4 \sin (0°) = 0$$

$$F_2 = F_3 = 180 \text{ kip} \qquad (b)$$

Finally, Eq. (E.8), Eq. (E.10), Eq. (a) and Eq. (b) enable us to write:

$$S_{vx} = - F_1 \cos (0°) + F_2 \cos (45°) + F_3 \cos (45°) - F_4 \cos (0°) = 0$$

$$F_4 = [F_2 + F_3] \cos (45°) - F_1 = (2)(180)(0.7071) - 144 = 110.6 \text{ kip} \qquad (c)$$

EXAMPLE E.10

Determine the magnitude and the direction of the force **F** in Fig. EXPL.10 if the box is lifted from the floor without swinging. The weight of the box is 27.0 kN.

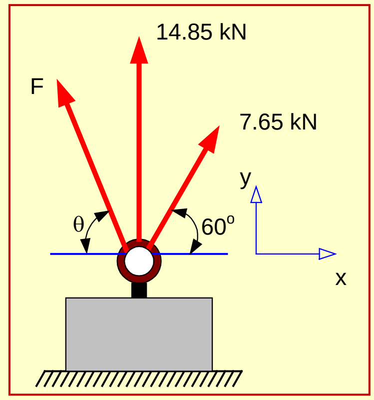


Fig. EXPL.10

Solution:

If the box is lifted without swinging the component of the vector sum of the three forces in the x direction must be zero, or $S_{vx} = 0$

From Eq. (E.8) and Eq. (E.10), we may write:

$$S_{vx} = (7.65) \cos (60°) - F \cos \theta = 0 \qquad \text{(a)}$$

$$F \cos \theta = 3.825 \text{ kN} \qquad \text{(b)}$$

For the box to lift from the floor the vertical sum of the three forces must equal the box's weight of 27.0 kN. This fact permits us to write:

$$S_{vy} = W = 27.0 = (7.65) \sin (60°) + 14.85 + F \sin \theta \qquad \text{(c)}$$

$$F \sin \theta = 5.525 \text{ kN} \qquad \text{(d)}$$

Dividing the results of Eq. (d) by Eq. (b) yields:

$$(F \sin \theta / F \cos \theta) = \tan \theta = (5.525)/(3.825) = 1.444 \qquad \text{(e)}$$

$$\theta = 55.30° \qquad \text{and} \qquad F = (5.525)/\sin (55.30°) = 6.720 \text{ kN} \qquad \text{(f)}$$

E.6 CONCURRENT AND COPLANAR FORCES

Concurrent and **coplanar** are descriptive words identifying certain vector systems. First, consider a concurrent vector system. When two or more forces (or vectors) act at a point Q, as illustrated in Fig. E.18, they are concurrent. When a body is subjected to a concurrent set of forces, it does not exhibit a tendency to rotate (i.e. the moments about the point Q are zero). Equilibrium of the body is determined by using only

$$\Sigma \mathbf{F} = 0 \qquad (1.2)$$

While the equilibrium of the forces given by Σ **F = 0** will prevent a body from translating (linear motion), this relation does not guarantee that the body will not rotate. To understand why bodies rotate, the concept of **moments** must be introduced. Moment vectors, **M**, cannot be produced by force systems that are concurrent. The equilibrium condition to prevent rotation of a body is given by:

$$\Sigma \mathbf{M} = 0 \qquad (E.13)$$

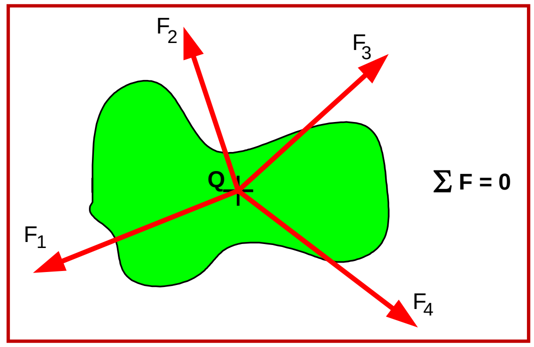


Fig. E.18 Forces $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$, and $\mathbf{F_4}$ are concurrent because they all are applied to point Q.

The relation $\Sigma\ \mathbf{M} = 0$ is satisfied automatically by concurrent force systems. We will discuss moments produced by forces in much more detail in Sections E.8 and E.9.

Coplanar forces (vectors) all lie in the same plane. If we have any two vectors, it is easy to show that they are coplanar regardless of their orientation. However, if we have three or more vectors, the vector system may or may not be coplanar. Examples of planar and non-planar vector systems are shown in Figs. E.19 and E.20, respectively.

In Fig. E.19a, the vectors **A**, **B**, and **C** all lie in the x-y plane, and the vector system is coplanar. However, this system is not concurrent. Vectors **A** and **C** both can be shifted along their lines of action to produce an intersection of their tail points, but it is impossible to shift vector **B** along its line of action to intersect at this point. It is evident that vector **B** is offset from this intersection point, it and produces a moment about this point.

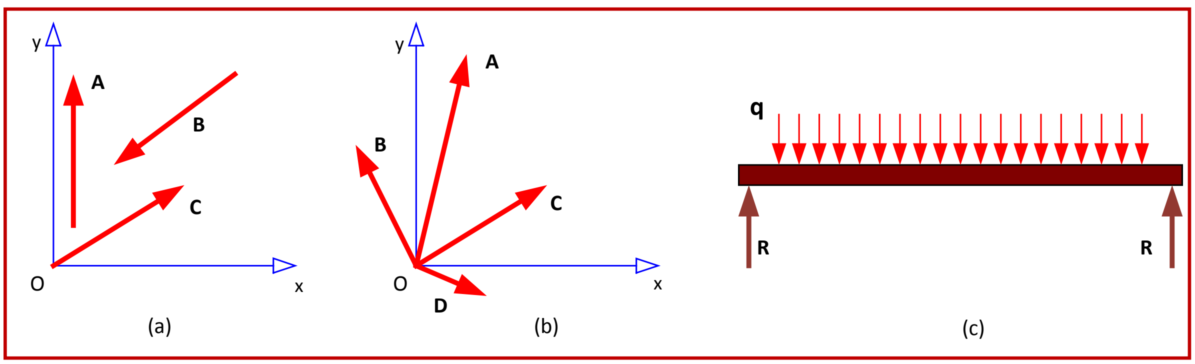


Fig. E.19 Examples of coplanar forces.

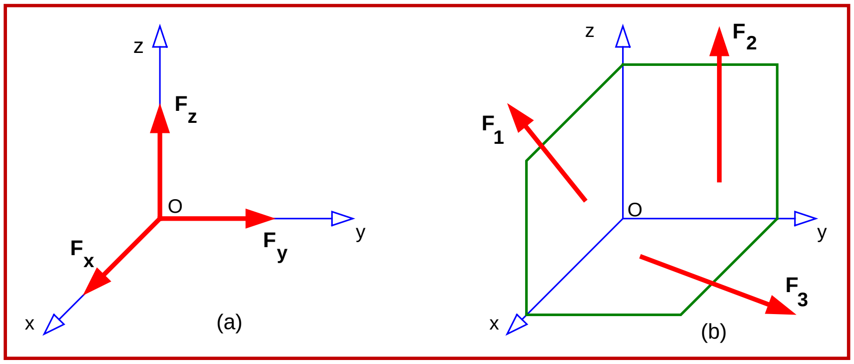
In Fig. E.19b, the vectors **A**, **B**, **C** and **D** all lie in the x-y plane, and the vector system is coplanar. Moreover, this system is concurrent because all four of the vectors intersect at point O. The force vectors in Fig. E.19c are coplanar, because both of the reaction forces **R** and the individual forces constituting the distributed load q all act in the plane of the beam. This system is not concurrent, because all of the force vectors are parallel, and cannot be shifted along their lines of action to intersect at a common point.

The three dimensional force systems, illustrated in Figs. E.20a and E.20b, are not coplanar. In Fig. E.20a, we show a Cartesian force system with vectors $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$, which coincide with the x, y and z axes, respectively. Because these three forces pass though the origin, the system is concurrent. In Fig. E.20b, the vector $\mathbf{F_1}$ lies in the x-z plane, $\mathbf{F_2}$ lies in the y-z plane and $\mathbf{F_3}$ lies in the x-y plane. Clearly, this system is non-coplanar and non-concurrent.

Classification of force vector systems is important because it indicates the number of equations of equilibrium that provide useful relations in solving for the forces and moments occurring in structural members. The useful equilibrium relations for the four possible classifications shown below, were presented previously in Chapter 2:

- Coplanar and concurrent.
- Coplanar and non-concurrent.
- Non-coplanar and concurrent
- Non-coplanar and non-concurrent.

Fig. E.20 Examples of non-coplanar vectors.



Three-dimensional force systems, like those illustrated in Fig. E.20, are referred to as **space forces**. They are more complex, because of the additional equations that must be satisfied to maintain a body in equilibrium, when it is subjected to a three-dimensional system of forces.

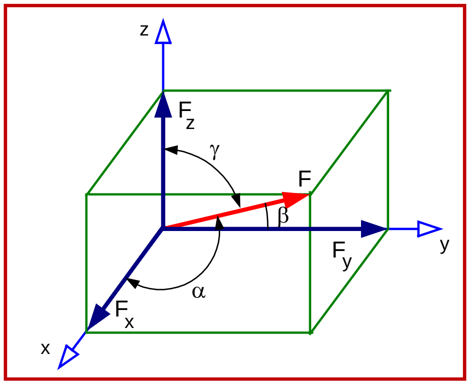
E.7 <u>SPACE FORCES</u>

When forces are coplanar, we place them on an x-y plane and apply the equilibrium equations to solve for the unknown forces acting on the structure under consideration. However, in some structures we must consider space forces, which cannot be completely characterized on an x–y plane. Consider the space force **F** shown in Fig. E.21, which is represented with Cartesian vectors $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$.

The space force **F** has both magnitude and direction. Its magnitude is a scalar quantity F, and its direction relative to a three-dimensional Cartesian coordinate system is given by the angles α, β and γ. These quantities, known as the coordinate direction angles, are measured from the line of action of **F** to the positive x, y and z-axes. The coordinate direction angles of the space vector F are given by:

$$\cos\alpha = \frac{F_x}{F} \qquad \cos\beta = \frac{F_y}{F} \qquad \cos\gamma = \frac{F_z}{F} \tag{E.14}$$

Fig. E.21 A space force F is shown with its equivalent Cartesian vectors $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$.



Also, recall the important identity for the coordinate direction angles that is given by:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{E.15}$$

From Eq. (E.14), it is evident that the magnitudes of the Cartesian components of the space force are given by:

$$F_x = F\cos\alpha \qquad F_y = F\cos\beta \qquad F_z = F\cos\gamma \tag{E.16}$$

The terms cos α, cos β and cos γ that appear in Eqs. (E.14) and Eq. (E.16) are called direction cosines of the vector **F**. Finally, the magnitude F of the vector **F** is given by the positive square root of the sum of the squares of its three components.

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{E.17}$$

EXAMPLE E.11

A space force with a magnitude of 3,200 N is positioned at the origin of a Cartesian coordinate system. The force is oriented so that the angles $\alpha = 45°$ and $\beta = 75°$. Determine the Cartesian components of the space force and the angle that it makes with the z-axis. Assume $\gamma \leq 90°$.

Solution:

From Eq. (E.15), we write:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \cos^2(45°) + \cos^2(75°) + \cos^2\gamma = 1 \tag{a}$$

This relation reduces to:

$$\cos^2\gamma = 1 - 0.5 - 0.06699 = 0.4330 \tag{b}$$

Solving for the angle γ yields:

$$\cos\gamma = 0.6580 \quad \Rightarrow \qquad \Rightarrow \qquad \gamma = 48.85° \tag{c}$$

Next, let's recall Eq. (E.16) and apply it to determine the Cartesian components of the space force.

$$F_x = F\cos\alpha \qquad F_y = F\cos\beta \qquad F_z = F\cos\gamma$$

$$F_x = (3{,}200)\cos(45°) \qquad F_y = (3{,}200)\cos(75°) \qquad F_z = (3{,}200)\cos(48.85°) \tag{d}$$

$$F_x = 2{,}263\text{ N} \qquad F_y = 828.2\text{ N} \qquad F_z = 2{,}106\text{ N}$$

We may check the accuracy of the result by employing Eq. (E.17).

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{e}$$

Substituting numerical values into this relation gives:

$$F = \sqrt{(2{,}263)^2 + (828.2)^2 + (2{,}106)^2} = \sqrt{1{,}024 \times 10^4} = 3{,}200 \text{ N} \qquad \text{(f)}$$

The results confirm the accuracy of the computations.

EXAMPLE E.12

A space force is represented by its Cartesian components as:

$F_x = 900$ lb $\qquad F_y = 1{,}224$ lb $\qquad F_z = 552$ lb

Determine the magnitude of the space force and its coordinate direction angles.

Solution:

Let's first determine the magnitude of the space force by using Eq. (E.17)

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(900)^2 + (1{,}224)^2 + (552)^2} = 1{,}616.4 \text{ lb} \qquad \text{(a)}$$

The coordinate direction angles are calculated from Eq. (E.14) as:

$$\cos\alpha = \frac{F_x}{F} = \frac{900}{1{,}616.4} = 0.5568$$
$$\cos\beta = \frac{F_y}{F} = \frac{1{,}224}{1{,}616.4} = 0.7572 \qquad \text{(b)}$$
$$\cos\gamma = \frac{F_z}{F} = \frac{552}{1{,}616.4} = 0.3415$$

Solving for the coordinate direction angles yields:

$\alpha = 56.17°$ $\qquad \beta = 40.78°$ $\qquad \gamma = 70.03°$ $\qquad$ (c)

Again we may check the accuracy of our calculations by using Eq. (E.15). Substituting the results for the coordinate direction angles into this relation yields:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (0.5568)^2 + (0.7572)^2 + (0.3415)^2 = 1.000 \qquad \text{(d)}$$

Because the identity for the coordinate direction angles is satisfied, we may conclude that the calculation was performed correctly.

E.7.1 Unit Vectors

Let's define a force vector **F** in terms of its unit vectors as shown below:

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \qquad \text{(E.18)}$$

where F_x, F_y, F_z are the magnitudes of the Cartesian force components, and
i, j, k are the Cartesian unit vectors in the x, y and z directions, respectively.

As their name implies, the unit vectors have a magnitude of unity. They provide directions along the x, y and z-axes in Eq. (E.18) for the Cartesian vectors $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$. The unit vectors **i**, **j** and **k** are illustrated in Fig. E.22. In many cases it is useful to employ unit vectors that are oriented in an arbitrary direction. For example consider a vector **F** as shown in Fig. E.23.

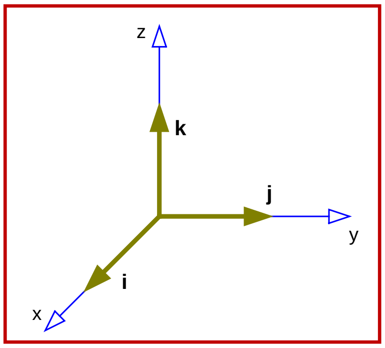


Fig. E.22 Unit vectors **i**, **j** and **k** oriented along the x, y and z axes, respectively.

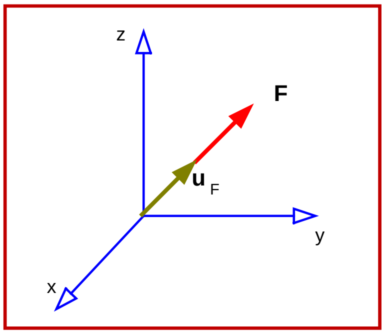


Fig. E.23 A unit vector $\mathbf{u}_F$ gives the direction of the vector **F**.

The direction of this unit vector is given by:

$$\mathbf{u}_F = \mathbf{F}/F \tag{E.19}$$

Hence, the vector **F** may be written as:

$$\mathbf{F} = F\,\mathbf{u}_F \tag{E.20}$$

A comparison of Eqs. (E.18) and (E.20) indicates that we may write the equation for a force using either its Cartesian unit vectors, which are directed along the x, y and z-axes, or by a single unit vector along the line of action of the force. The direction of the force may also be expressed using either the unit vector $\mathbf{u}_F$ or the coordinate direction angles. This fact implies a relation between these quantities. To show this relation, we substitute Eq. (E.18) into Eq. (E.19) to obtain:

$$\mathbf{u}_F = \mathbf{F}/F = (F_x/F)\mathbf{i} + (F_y/F)\mathbf{j} + (F_z/F)\mathbf{k} \tag{E.21}$$

Substituting Eq. (E.14) into Eq. (E.21) yields:

$$\mathbf{u}_F = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} \tag{E.22}$$

EXAMPLE E.13

A space force is represented by its Cartesian components as:

$F_x = 602$ N $\qquad F_y = 334$ N $\qquad F_z = 818$ N

Write an expression for the force vector **F** in terms of the Cartesian unit vectors and determine the magnitude of the force F.

Solution:

Using Eq. (E.18), we write:

$$\mathbf{F} = (602\,\mathbf{i} + 334\,\mathbf{j} + 818\,\mathbf{k})\text{ N} \qquad \text{(a)}$$

The magnitude F is calculated from Eq. (E.17) as:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(602)^2 + (334)^2 + (818)^2} = 1{,}069\text{ N} \qquad \text{(b)}$$

EXAMPLE E.14

Determine the unit vector describing the direction of the space force specified in Example E.13. Also, calculate the coordinate direction angles for this force vector.

Solution:

To determine the unit vector $\mathbf{u}_F$, we substitute the results from Example E.13 into Eq. (E.21) to obtain:

$$\mathbf{u}_F = \mathbf{F}/F = (F_x/F)\mathbf{i} + (F_y/F)\mathbf{j} + (F_z/F)\mathbf{k} = (602/1{,}069)\,\mathbf{i} + (334/1{,}069)\,\mathbf{j} + (818/\,1{,}069)\,\mathbf{k}$$

$$\mathbf{u}_F = 0.5631\,\mathbf{i} + 0.3124\,\mathbf{j} + 0.7652\,\mathbf{k} \qquad \text{(a)}$$

The coordinate direction angles for the space force are determined from Eq. (E.14) as:

$$\cos\alpha = \frac{F_x}{F} = \frac{602}{1{,}069} = 0.5631$$

$$\cos\beta = \frac{F_y}{F} = \frac{334}{1{,}069} = 0.3124 \qquad \text{(b)}$$

$$\cos\gamma = \frac{F_z}{F} = \frac{818}{1{,}069} = 0.7652$$

Solving for the coordinate direction angles gives:

$\alpha = 55.73°$ $\qquad \beta = 71.80°$ $\qquad \gamma = 40.08°$ $\qquad$ (c)

EXAMPLE E.15

Express the space force **F**, shown in Fig. EXPL.15, in terms of its Cartesian components.

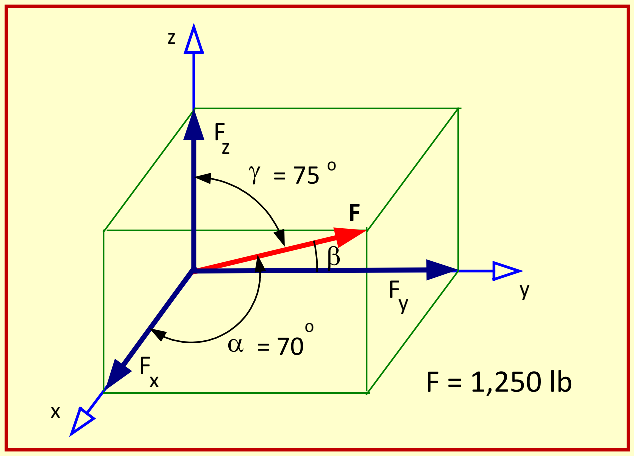


Fig. EXPL.15

Solution:

Because only two of the direction cosine angles have been specified, we begin by using Eq. (E.15) to determine the unknown angle β.

$$\cos^2\beta = 1 - \cos^2\alpha - \cos^2\gamma = 1 - \cos^2(70°) - \cos^2(75°) \qquad \text{(a)}$$

$$\cos^2\beta = 1 - 0.1170 - 0.06699 = 0.8160 \qquad \text{(b)}$$

$$\cos\beta = \pm\, 0.9033 \qquad \Rightarrow \qquad \Rightarrow \qquad \beta = 25.40^{o} \text{ or } 154.6° \qquad \text{(c)}$$

By inspection of Fig. EXPL.15, it is evident that β is an acute angle; therefore:

$$\beta = 25.40° \qquad \text{(d)}$$

Next, employ Eq. (E.16) to determine the Cartesian components of the space force **F** as:

$$F_x = F\cos\alpha = (1{,}250)\cos(70.0°) = 427.5 \text{ lb}$$

$$F_y = F\cos\beta = (1{,}250)\cos(25.4°) = 1{,}129 \text{ lb} \qquad \text{(e)}$$

$$F_z = F\cos\gamma = (1{,}250)\cos(75.0°) = 323.5 \text{ lb}$$

Finally, let's check the accuracy of these results by using Eq. (E.17).

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(427.5)^2 + (1{,}129)^2 + (323.5)^2} = 1{,}250 \text{ lb} \qquad \text{(f)}$$

The result of 1,250 lb for the magnitude of F is correct and it verifies the accuracy of the calculation of the force components.

E.7.2 <u>Position Vectors</u>

A position vector is a fixed vector in space. It is used to locate one point in space relative to another. A position vector is similar to a force vector in that it has both magnitude and direction. An example of a position vector is presented in Fig. E.24. In this case, the position vector **r** locates a point A (x, y, z) relative to the origin of an Oxyz coordinate system.

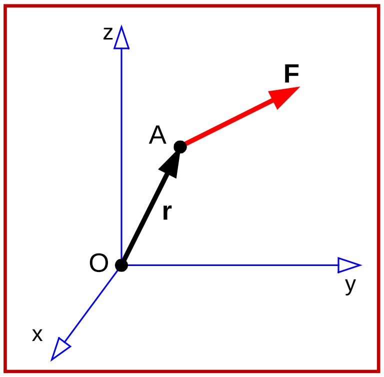


Fig. E.24 The position vector **r** locates the point of application of **F** relative to the coordinate's origin.

The position vector in Fig. E.24 is expressed as:

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \qquad \text{(E.23)}$$

The coordinate direction angles of the position vector **r** are:

$$\cos\alpha = x/r \qquad \cos\beta = y/r \qquad \cos\gamma = z/r \qquad \text{(E.24)}$$

Finally, the magnitude of the position vector r is given by:

$$r = [x^2 + y^2 + z^2]^{1/2} \qquad \text{(E.25)}$$

EXAMPLE E.16

A force vector **F** is applied to a rectangular space frame at point A, as shown in Fig. EXPL.16. Write the equation for the position vector **r** locating the point of application relative to the origin of the Cartesian coordinates. Also determine the magnitude and direction of the position vector **r**.

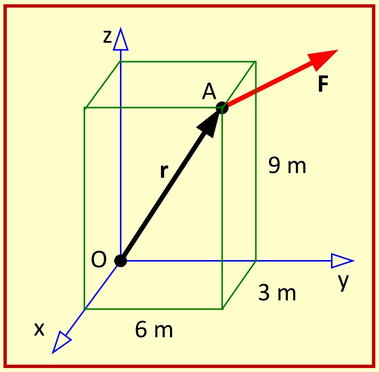


Fig. EXPL.16

Solution:

Selecting the dimensions of the rectangular space frame from Fig. EXPL.16 and using Eq. (E.23) enables us to write:

$$\mathbf{r} = (3\,\mathbf{i} + 6\,\mathbf{j} + 9\,\mathbf{k})\text{ m} \qquad \text{(a)}$$

The magnitude of the position vector is given by substituting the coordinates into Eq. (E.25):

$$\mathbf{r} = \sqrt{x^2 + y^2 + z^2} = \sqrt{(3)^2 + (6)^2 + (9)^2} = 11.225 \text{ m} \qquad \text{(b)}$$

Next, we employ Eq. (E.24) to determine the coordinate direction angles for the vector **r**.

$$\cos\alpha = x/r = (3)/(11.225) = 0.2673 \quad \Rightarrow \quad \alpha = 74.50°$$

$$\cos\beta = y/r = (6)/(11.225) = 0.5345 \quad \Rightarrow \quad \beta = 57.69° \qquad \text{(c)}$$

$$\cos\gamma = z/r = (9)/(11.225) = 0.8018 \quad \Rightarrow \quad \gamma = 36.70°$$

Let's check the accuracy of the calculations by using Eq. (E.15).

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (0.2673)^2 + (0.5345)^2 + (0.8018)^2 = 1.000 \qquad \text{(d)}$$

The results for the direction cosine angles check.

Let's consider a more general case for a position vector, which locates point B relative to point A. In this case neither point A nor B are located at the origin of the coordinate system. The position vector for this situation is presented in Fig. E.25.

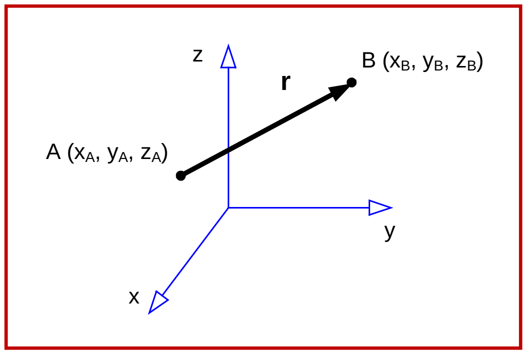


Fig. E.25 A position vector r that locates point B relative to point A.

The **i**, **j** and **k** components of the position vector **r** in Fig. E.25 are written by taking the coordinates of the tip at B (x_B, y_B, z_B) and subtracting the coordinates of the tail at A (x_A, y_A, z_A), as indicated below:

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} \qquad \text{(E.26)}$$

The coordinate direction angles of the position vector **r** are:

$$\cos\alpha = (x_B - x_A)/r \qquad \cos\beta = (y_B - y_A)/r \qquad \cos\gamma = (z_B - z_A)/r \qquad \text{(E.27)}$$

These coordinate direction angles are measured from a local coordinate system positioned at point A, which is the tail of the position vector **r**. The magnitude of the position vector in Fig. E.25 is given by:

$$r = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \qquad \text{(E.28)}$$

We may also specify the direction of **r** by writing the expression for the unit vector $\mathbf{u_r}$. Let's adapt Eq. (E.21) and write:

$$\mathbf{u}_r = \mathbf{r}/r = (r_x/r)\mathbf{i} + (r_y/r)\mathbf{j} + (r_z/r)\mathbf{k} \qquad \text{(E.29)}$$

EXAMPLE E.17

Determine the magnitude and direction of the position vector that extends from point A to B in Fig. EXPL.17. The coordinates of A and B, expressed in m, are given by A = (2, 1, –4) and B = (–3, 5, 4).

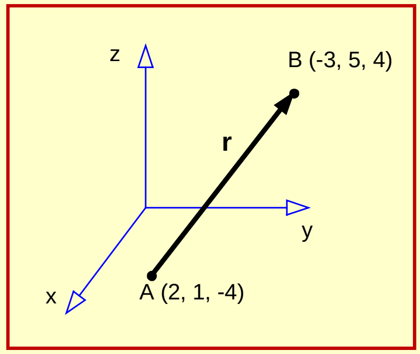


Fig. EXPL.17

Solution:

Let's write the expression for r by using Eq. (E.26).

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} = (-3 - 2)\,\mathbf{i} + (5 - 1)\,\mathbf{j} + (4 + 4)\,\mathbf{k}$$

$$\mathbf{r} = (-5\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k})\text{ m} \qquad \text{(a)}$$

The magnitude of the position vector is given by Eq. (E.28):

$$\mathbf{r} = \sqrt{(-5)^2 + (4)^2 + (8)^2} = 10.25\text{ m} \qquad \text{(b)}$$

The coordinate direction angles are computed from Eq. (E.27).

$$\cos\alpha = (x_B - x_A)/r = (-5)/(10.25) = -0.4878 \;\Rightarrow\; \alpha = 119.2°$$

$$\cos\beta = (y_B - y_A)/r = (4)/(10.25) = 0.3902 \;\Rightarrow\; \beta = 67.03° \qquad \text{(c)}$$

$$\cos\gamma = (z_B - z_A)/r = (8)/(10.25) = 0.7805 \;\Rightarrow\; \gamma = 38.69°$$

The direction of **r** is specified by determining the unit vector $\mathbf{u_r}$ from Eq. (E.29):

$$\mathbf{u_r} = \mathbf{r}/r = (-5\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k})/(10.25) = -0.4878\,\mathbf{i} + 0.3902\,\mathbf{j} + 0.7805\,\mathbf{k} \qquad \text{(d)}$$

Notice that the coefficients of the unit vectors are identical to the results for the coordinate direction cosines, as shown previously in Eq. (E.22).

EXAMPLE E.18

A helicopter is attempting to tow a disabled supply truck along a road parallel to the y-axis in Fig. EXPL.18. Unfortunately, the pilot is not properly aligned for the task with an initial position given by the coordinates x = – 15 ft, y = – 10 ft, and z = 50 ft relative to the hitch point on the truck. If a force component F_y of 1,750 lb is required to tow the truck, determine the force that the helicopter must apply to the towline.

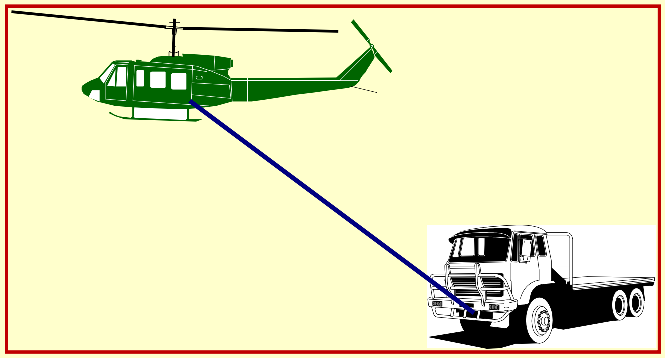


Fig. EXPL.18

Solution:

Prepare a drawing of the truck tow point and the helicopter hitch point relative to a three-dimensional Cartesian coordinate system as shown in Fig. EXPL.18a. We use the tow point on the truck as the origin of the Cartesian coordinates.

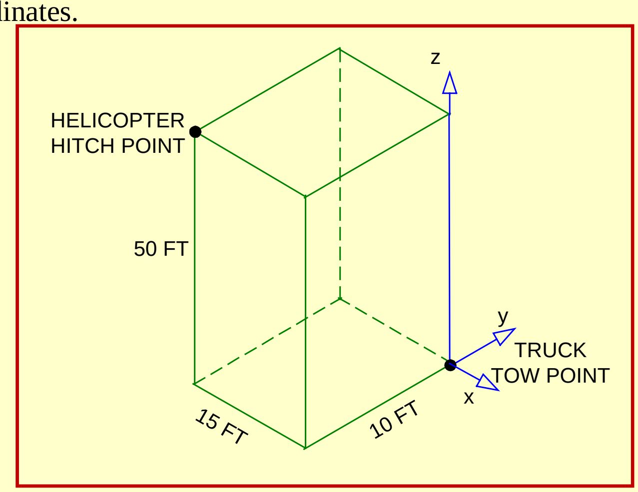


Fig. EXPL.18a

Let's begin by determining the coordinate direction angles of the force that the helicopter will exert on the truck. The magnitude of position vector from the truck tow point to the helicopter hitch point is given by Eq. (E.25).

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-15)^2 + (-10)^2 + (50)^2} = 53.15 \text{ ft} \qquad \text{(a)}$$

The coordinate direction angles are given by Eq. (E.24) as indicated below:

$$\cos\alpha = \frac{x}{r} = \frac{-15}{53.15} = -0.2822 \quad \Rightarrow \alpha = 106.4°$$

$$\cos\beta = \frac{y}{r} = \frac{-10}{53.15} = -0.1881 \quad \Rightarrow \beta = 100.8° \qquad \text{(b)}$$

$$\cos\gamma = \frac{z}{r} = \frac{50}{53.15} = 0.9407 \quad \Rightarrow \gamma = 19.83°$$

Because the force and the rope are along the same line of action, the coordinate direction angles are identical for the position vector **r** and the force **F**. Recognizing this fact permits us to solve for the magnitude of the force necessary to tow the truck by using Eq. (E.16).

$$F = F_y/\cos\beta \qquad (c)$$

$$F = (-1{,}750)/(-0.1881) = +\,9{,}304 \text{ lb} \qquad (d)$$

The negative sign for the force component F_y is used to recognize that the force is being applied in the negative y direction.

EXAMPLE E.19

A large cylindrical pontoon is moored at a dock with a line attached to it at a mooring cleat, as indicated in Fig. EXPL.19. If the cleat is at a position on the pontoon given by coordinates x = 3 m, y = – 8 m and z = – 2 m, determine the tension in the mooring line if the wind and current are producing a drag force on the pontoon of 715 N in the x direction.

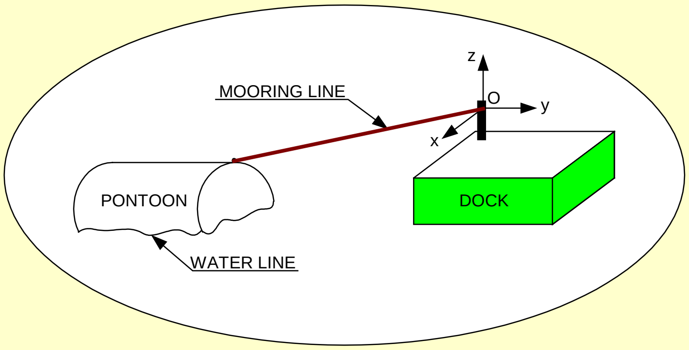


Fig. EXPL.19

Solution:

This situation is similar to the previous example where a rope or line is used to apply a force. In these situations, the position vector describing the orientation of the line and the force coincide. Because we know the coordinates of the rope, we can determine its length and orientation from Eqs. (E.24) and (E.25). The length of the position vector from the dock to the cleat is:

$$r = \sqrt{(3)^2 + (-8)^2 + (-2)^2} = 8.775 \text{ m} \qquad (a)$$

The coordinate direction angles are given by modifying Eq. (E.24) as indicated below:

$$\begin{aligned}
\cos\alpha &= \frac{r_x}{r} = \frac{3}{8.775} = 0.3419 &&\Rightarrow \alpha = 70.01^\circ \\
\cos\beta &= \frac{r_y}{r} = \frac{-8}{8.775} = -0.9117 &&\Rightarrow \beta = 155.7^\circ \qquad (b) \\
\cos\gamma &= \frac{r_z}{r} = \frac{-2}{8.775} = -0.2279 &&\Rightarrow \gamma = 103.2^\circ
\end{aligned}$$

The magnitude of the force on the mooring line is determined from Eq. (E.16) as:

$$F = F_x/\cos\alpha = (715)/(0.3419) = 2{,}091 \text{ N} \qquad (c)$$

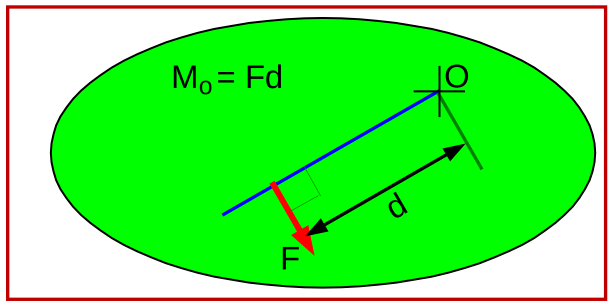
E.8 MOMENTS

If you have ever used a wrench or screwdriver to tighten a bolt or screw, you have generated a moment. A moment M_o about a point O is produced when a force F, as shown in Fig. E.26, is applied in such a manner that it tends to cause a body to rotate about this point. The magnitude of a moment produced by a force is dependent on the location of point O, and is given by:

$$M_o = F\ d \tag{E.30}$$

where d is the perpendicular distance from the point O to the line of action of F.

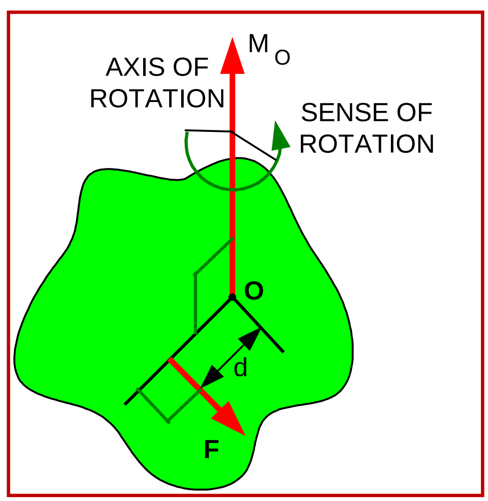
Fig. E.26 A moment produced by a force depends on the position of point O.



The units of a moment $\mathbf{M_o}$ are given as N-m or ft-lb, as was stated in Table 1.2. $\mathbf{M_o}$ is a vector quantity; hence, it must be specified with both magnitude and direction. Its magnitude is given by Eq. (E.30), and the direction of the vector $\mathbf{M_o}$ is perpendicular to the plane in which both F and d lie. As shown in Fig. E.27, the moment $\mathbf{M_o}$ has a sense of direction. In this illustration, the moment $\mathbf{M_o}$, when viewed from above, tends to rotate the body in a **counterclockwise** direction and is **positive**.

To determine the sign of the moment, we use the **right hand rule**. In applying this rule, place the palm of your right hand along the axis of rotation and point your fingers in the direction of the force and rotate your hand. If the direction of the force causes you to rotate **counterclockwise**, with your thumbs pointing up from the plane in which both F and d lie, then the moment is **positive**. However, if you must rotate your hand **clockwise**, with your thumb pointing downward, the moment is **negative**.

Fig. E.27 A graphic illustration of the moment $\mathbf{M_o}$ as a positive vector quantity.



EXAMPLE E.20

A hexagonal headed bolt is tightened with a wrench, as shown in Fig. EXPL.20. A 32-lb force is applied to the handle of the wrench to produce a moment (torque). If the distance from the centerline of the bolt to the point of application of the force is d = 9 in., find the applied torque.

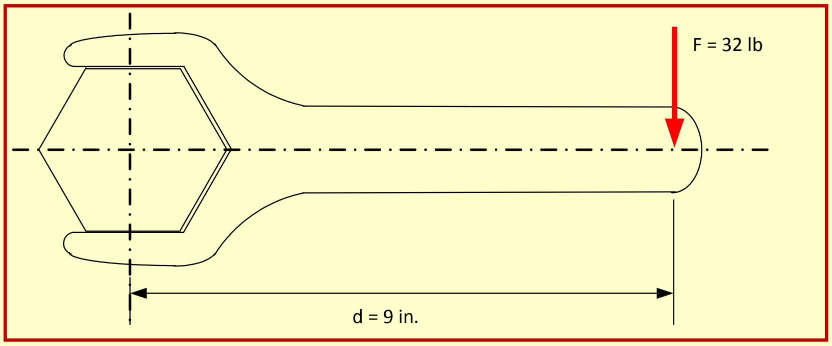


Fig. EXPL.20

Solution:

From Eq. (E.30), we write

$$M = F\,d = (32)(9) = 288 \text{ in-lb} = 24.0 \text{ ft-lb} \qquad \text{(a)}$$

We consider this moment to be **negative** because it tends to produce a **clockwise** rotation about the head of the bolt.

EXAMPLE E.21

Suppose a force of 88 N is applied to the wrench, as shown in Fig. EXPL.21. Determine the torque (moment) applied to the hex headed bolt.

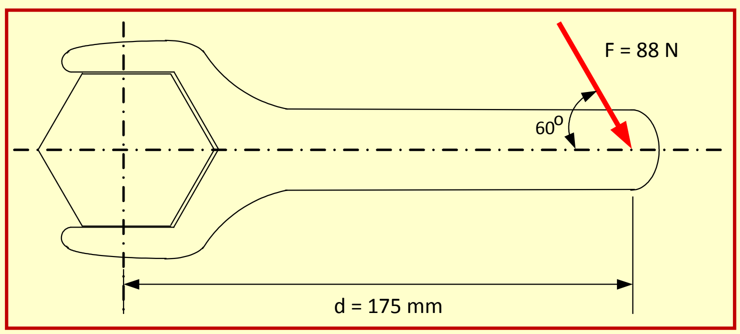


Fig. EXPL.21

Solution:

Let's resolve the 88 N force into two components — one perpendicular to the axis of the wrench and the other parallel.

The component of force parallel to the axis of the wrench is obtained by:

$$F_{=} = (88) \cos (60°) = 44.0 \text{ N} \qquad \text{(a)}$$

Note that $F_{=}$ does not produce a moment (torque) on the bolt because its moment arm d = 0. The component perpendicular to the axis of the wrench is given by:

$$F_{\perp} = (88) \sin (60°) = 76.21 \text{ N} \qquad \text{(b)}$$

To determine the moment that this component of force produces on the head of the bolt, we use Eq. (E.30) and write:

$$M = F\,d = (76.21)(0.175) = 13.34 \text{ N-m} \qquad \text{(c)}$$

Again, this moment is **negative** because it tends to produce a **clockwise** rotation about the head of the bolt.

E.9 VECTOR MECHANICS

E.9.1 Expressing Forces and Moments as Vectors

When we encounter three-dimensional structures, the analysis becomes more complex often involving six equations of equilibrium and several free body diagrams. Visualization of the structure and force components often becomes more difficult. To alleviate the complexity and reduce visualization difficulties, we often use a vector mechanics (algebra) approach. When we express each force **F** or moment **M** in a complete vector representation, the equilibrium relations reduce to two vector equations:

$$\Sigma\mathbf{F} = \mathbf{0} \qquad \text{and} \qquad \Sigma\mathbf{M} = \mathbf{0}$$

The **Σ** symbol implies that we sum all of the forces, the applied moments, and all of the moments produced by the forces acting on the structure or one of its members. Hence, we write these relations as:

$$\Sigma\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \ldots\ldots\ldots + \mathbf{F_n} = \mathbf{0} \qquad \text{(E.31)}$$

$$\Sigma\mathbf{M} = \mathbf{M_1} + \mathbf{M_2} + \ldots\ldots\ldots + \mathbf{M_n} = \mathbf{0} \qquad \text{(E.32)}$$

where forces 1, 2, 3 **…….** n act on the structure.

Next set the sums to zero assuming the body is in equilibrium. Then use Cartesian unit vectors in expressing the force vectors $\mathbf{F_1}$, $\mathbf{F_2}$, to $\mathbf{F_n}$ as:

$$\begin{aligned}
\mathbf{F_1} &= F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j} + F_{1z}\,\mathbf{k} \\
\mathbf{F_2} &= F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j} + F_{2z}\,\mathbf{k} \qquad \text{(E.33)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mathbf{F_n} &= F_{nx}\,\mathbf{i} + F_{ny}\,\mathbf{j} + F_{nz}\,\mathbf{k}
\end{aligned}$$

where F_{1x}, F_{1y}, F_{1z}, etc. are the magnitudes of the Cartesian force components.

Substituting Eq. (E.33) into Eq. (E.31) yields:

$$\Sigma\mathbf{F} = (F_{1x} + F_{2x} + \ldots + F_{nx})\mathbf{i} + (F_{1y} + F_{2y} + \ldots + F_{ny})\mathbf{j} + (F_{1z} + F_{2z} + \ldots + F_{nz})\mathbf{k} = 0 \qquad \text{(E.34)}$$

Inspection of this relation shows that:

$$F_{1x} + F_{2x} + \ldots + F_{nx} = \Sigma F_x = 0$$

$$F_{1y} + F_{2y} + \ldots + F_{ny} = \Sigma F_y = 0 \qquad \text{(E.35)}$$

$$F_{1z} + F_{2z} + \ldots + F_{nz} = \Sigma F_z = 0$$

These three relations represent the equilibrium equations expressed in terms of scalar force components. We will employ these relations throughout most of this text. With the vector mechanics approach, they are combined into a single relation given by Eq. (E.34).

EXAMPLE E.22

A space force **F** with a magnitude of 925 lb has coordinate direction angles of $\alpha = 45^o$ and $\beta = 45^o$. Write a vector equation for **F** and prepare a drawing showing the force vector in a three-dimensional Cartesian coordinate system.

Solution:

The specification of the force **F** in the problem statement is not complete because the coordinate direction angle γ is not given. We may determine angle γ by recalling Eq. (E.15).

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \cos^2 (45°) + \cos^2 (45°) + \cos^2 \gamma = 1$$

$$0.5 + 0.5 + \cos^2 \gamma = 1 \quad \Rightarrow \quad \cos^2 \gamma = 0 \quad \Rightarrow \quad \gamma = 90° \qquad \text{(a)}$$

The Cartesian components of the force vector are given by Eq. (E.16) as:

$$F_x = F \cos \alpha \qquad F_y = F \cos \beta \qquad F_z = F \cos \gamma \qquad \text{(b)}$$

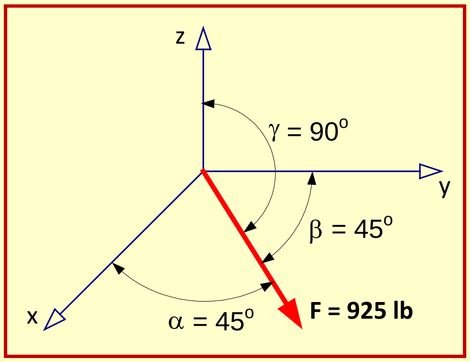
$$F_x = (925) \cos (45°) = 654.1 \text{ lb} \qquad F_y = (925) \cos (45°) = 654.1 \text{ lb} \qquad F_z = (925) \cos (90°) = 0$$

Next, we may write the vector equation for the 925 lb force as:

$$\mathbf{F} = (654.1\ \mathbf{i} + 654.1\ \mathbf{j}) \text{ lb} \qquad \text{(c)}$$

Note, the multiplier of the **k** unit vector is absent from this result because $F_z = 0$. Finally, the force vector **F** is shown in the x – y plane in Fig. EXPL.22.

Fig. EXPL.22



EXAMPLE E.23

A space force $\mathbf{F_1}$ with a magnitude of 25 kN has coordinate direction angles of $\alpha = 60^o$, $\beta = 60^o$, and $\gamma = 135°$. A second space force $\mathbf{F}_2$ with a magnitude of 15 kN has coordinate direction angles of $\alpha = 90^o$, $\beta = 60^o$, and $\gamma = 30°$. Write a vector equation for the summation of these two space forces.

Solution:

The summation of $\mathbf{F_1}$ and $\mathbf{F_2}$ is given by Eq. (E.34) as:

$$\Sigma\mathbf{F} = (F_{1x} + F_{2x})\mathbf{i} + (F_{1y} + F_{2y})\mathbf{j} + (F_{1z} + F_{2z})\mathbf{k} \quad \text{(a)}$$

Let's first determine the Cartesian components of both space forces from Eq. (E.16):

$$F_{xi} = F_i \cos \alpha_i \qquad F_{yi} = F_i \cos \beta_i \qquad F_{zi} = F_i \cos \gamma_i \quad \text{(b)}$$

$$F_{x1} = (25) \cos (60°) = 12.5 \text{ kN} \qquad F_{x2} = (15) \cos (90°) = 0 \text{ kN}$$

$$F_{y1} = (25) \cos (60°) = 12.5 \text{ kN} \qquad F_{y2} = (15) \cos (60°) = 7.5 \text{ kN}$$

$$F_{z1} = (25) \cos (135°) = -17.68 \text{ kN} \qquad F_{z2} = (15) \cos (30°) = 12.99 \text{ kN}$$

We then substitute the values of the components in Eq. (a) to obtain:

$$\sum\mathbf{F} = (12.5 + 0)\mathbf{i} + (12.5 + 7.5)\mathbf{j} + (-17.68 + 12.99)\mathbf{k}$$

$$\sum\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} = (12.5\ \mathbf{i} + 20.0\ \mathbf{j} - 4.69\ \mathbf{k}) \text{ kN} \quad \text{(c)}$$

While we have summed forces in Example E.23, the sum has not been set to zero, as shown in Eq. (E.34). The sum of the forces acting on a body is zero, when the body is at rest or moving at a constant velocity. In this example problem, we have not considered equilibrium, because our purpose was to demonstrate the technique for adding together two vectors.

E.9.2 The Vector Dot Product

We have determined components of force vectors by using Eqs. (E.8) and (E.16), in several examples. These relations are based on a trigonometric approach where the vector and its components are easy to visualize. However, in some three-dimensional problems, visualization is more difficult. In these instances, the use of vector algebra for determining components of force and moment vectors is usually more efficient. To introduce the application of vector algebra, two different techniques for vector multiplication — the dot product and the cross product — are described. The dot product is useful in determining the component of a force along some arbitrary line or the angle between two lines. The cross product is used to determine the moment due to a vector force about some arbitrary point.

Let's define the dot product of vectors **A** and **B,** as shown below:

$$\mathbf{A} \bullet \mathbf{B} = AB \cos \theta \quad \text{(E.36)}$$

where the angle θ is defined in Fig. E.28.

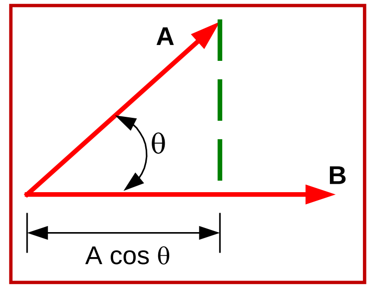


Fig. E.28 Vectors A and B and the angle θ.

The projection of the vector **A,** as a dimension measured along vector **B**, is shown in Fig. E.28. The vector dot product **A • B** is represented by the projection (A cos θ) multiplied by the magnitude B. The dot product of two vectors is a scalar quantity with its unit dependent on the product of the units of A and B. As a scalar quantity **A • B** has a magnitude, but not direction. We cannot draw an arrow to represent it, because it does not have an orientation.

Next let's consider the dot product of different pairs of unit vectors. It is easy to show that:

$$\mathbf{i} \bullet \mathbf{i} = (1)(1)\cos(0°) = 1 \qquad \text{and} \qquad \mathbf{i} \bullet \mathbf{j} = (1)(1)\cos(90°) = 0.$$

We list all possible combinations of the dot product of two Cartesian unit vectors below:

$$\begin{array}{lll} \mathbf{i} \bullet \mathbf{i} = 1 & \mathbf{i} \bullet \mathbf{j} = 0 & \mathbf{i} \bullet \mathbf{k} = 0 \\ \mathbf{j} \bullet \mathbf{j} = 1 & \mathbf{j} \bullet \mathbf{k} = 0 & \mathbf{j} \bullet \mathbf{i} = 0 \\ \mathbf{k} \bullet \mathbf{k} = 1 & \mathbf{k} \bullet \mathbf{i} = 0 & \mathbf{k} \bullet \mathbf{j} = 0 \end{array} \qquad \text{(E.37)}$$

Suppose we represent each of the two independent space vectors **A** and **B** with their Cartesian components:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

$$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k} \qquad \text{(a)}$$

Then the dot product of these two space vectors is given by:

$$\mathbf{A} \bullet \mathbf{B} = (A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}) \bullet (B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}) \qquad \text{(b)}$$

With the dot product relations for the unit vectors given in Eq. (E.37), it is evident that:

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{(E.38)}$$

As expected, the result in Eq. (E.38) is a scalar quantity. When we have two arbitrary vectors represented in vector format, the dot product is the sum of the product of their Cartesian components.

The dot product is a valuable analysis tool that may be used in two different applications:

1. To determine the angle θ between two space vectors. From Eq. (E.36) it is clear that:

$$\theta = \cos^{-1}[(\mathbf{A} \bullet \mathbf{B})/(AB)] \qquad \text{(E.39)}$$

2. To determine the component of a vector quantity along any line, define the direction of this line with a unit vector **u**. Then the component of vector **A** along this line is a scalar quantity given by:

$$A_u = \mathbf{A} \bullet \mathbf{u} \qquad \text{(E.40)}$$

Let's consider two examples to demonstrate the techniques involved in utilizing these tools.

EXAMPLE E.24

Suppose we have two space forces specified in vector format as:

$$\mathbf{F_1} = (6\,\mathbf{i} + 4\,\mathbf{j} - 1.5\,\mathbf{k})\text{ lb} \qquad \text{and} \qquad \mathbf{F_2} = (-3\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k})\text{ lb}$$

Determine the angle between the two forces.

Solution:

First determine the magnitude of each force from Eq. (E.17).

$$F_1 = [(6)^2 + (4)^2 + (-1.5)^2]^{1/2} = 7.365\text{ lb}$$

$$F_2 = [(-3)^2 + (2)^2 + (-1)^2]^{1/2} = 3.742\text{ lb} \qquad \text{(a)}$$

Next, use Eq. (E.38) to solve for $\mathbf{F_1} \bullet \mathbf{F_2}$ as:

$$\mathbf{F_1} \bullet \mathbf{F_2} = F_{1x}\,F_{2x} + F_{1y}\,F_{2y} + F_{1z}\,F_{2z} = (6)(-3) + (4)(2) + (-1.5)(-1) = -8.50\text{ lb}^2 \qquad \text{(b)}$$

Finally, apply Eq. (E.39) to determine the angle θ between the two forces.

$$\theta = \cos^{-1}[(\mathbf{F_1} \bullet \mathbf{F_2})/(F_1F_2)] = \cos^{-1}\{(-8.50)/[(7.365)(3.742)]\} = \cos^{-1}\{-0.3084\} = 108.0°$$

It is a very difficult exercise in solid geometry to determine the angle between two arbitrary lines. With the use of the dot product as a vector tool, the task becomes much easier.

EXAMPLE E.25

Suppose we have the same two space forces as in Example E.24 specified in vector format as:

$$\mathbf{F_1} = (6\,\mathbf{i} + 4\,\mathbf{j} - 1.5\,\mathbf{k})\text{ lb} \qquad \text{and} \qquad \mathbf{F_2} = (-3\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k})\text{ lb}$$

Determine the magnitude of a single force component produced by these two forces if it is directed along a line of action, which lies in the x – y plane making an angle of 45° with the x-axis.

Solution:

It is evident that we will employ Eq. (E.40) to determine the magnitude of this force component. The difficulty we first encounter is to write an expression for the unit vector giving the direction of the line in the x – y plane. It is easy to write a vector describing a line in the x – y plane with a 45° orientation as:

$$\mathbf{v} = \mathbf{i} + \mathbf{j} \qquad \text{(a)}$$

However, the magnitude of this vector is $(2)^{1/2}$; hence, it is not a unit vector. To convert vector $\mathbf{v}$ to a unit vector $\mathbf{u_v}$, we can utilize Eq. (E.29) as:

$$\mathbf{u_v} = \mathbf{v}/v = (\mathbf{i} + \mathbf{j})/(2)^{1/2} = 0.7071\,\mathbf{i} + 0.7071\,\mathbf{j} \qquad \text{(b)}$$

Substituting Eq. (b) into Eq. (E.40) gives:

$$F_{1u} = \mathbf{F_1} \bullet \mathbf{u} \qquad \text{and} \qquad F_{2u} = \mathbf{F_2} \bullet \mathbf{u} \tag{c}$$

$$F_{1u} = \mathbf{F_1} \bullet \mathbf{u} = (6\,\mathbf{i} + 4\,\mathbf{j} - 1.5\,\mathbf{k})\bullet(0.7071\,\mathbf{i} + 0.7071\,\mathbf{j}) = 7.071 \text{ lb}$$

$$F_{2u} = \mathbf{F_2} \bullet \mathbf{u} = (-3\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k})\bullet(0.7071\,\mathbf{i} + 0.7071\,\mathbf{j}) = -0.7071 \text{ lb} \tag{d}$$

$$F_u = F_{1u} + F_{2u} = 7.071 - 0.7071 = 6.364 \text{ lb} \tag{e}$$

Projecting forces and moments, both vector quantities, onto structural components that are oriented in arbitrary directions by using the properties of a dot product is a useful technique.

E.9.3 The Vector Cross Product

Let's define the mathematical form of the vector cross product as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin \theta)\mathbf{u} \tag{E.41}$$

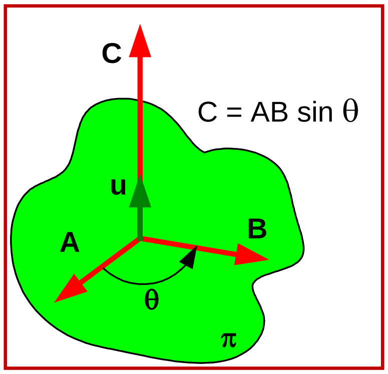
The cross product of vectors **A** and **B** yields a vector quantity **C**. Multiplying the product of the magnitudes of A and B by the sin θ gives the magnitude of C. The direction of vector **C** is perpendicular to the plane in which **A** and **B** lie. The sense of **C** is determined by the right hand rule: when curling the fingers of the right hand from **A** toward (cross) **B** with the normal to the plane π along the palm of your hand, the thumb points in the direction of vector **C**. A graphic representation of **C**, the cross product of vectors **A** and **B** as defined in Eq. (E.41), is presented in Fig. E.29.

Moments are a product of a force times a distance. In our previous discussion, we were always careful to define the distance as the perpendicular distance from the line of action of the force to the point about which the moments were determined. The definition of the moment does not change, but with the vector algebra approach, we find it helpful to define the moment in terms of a vector cross product as:

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} \tag{E.42}$$

where **r** is the position vector that locates point Q relative to the origin of a Cartesian coordinate system, as shown in Fig. E.30

Fig. E.29 Vector **C** is the cross product **A** × **B**.



The moment $\mathbf{M_O}$, relative to the origin O of a Cartesian coordinate system, is due to the force **F** that is applied at point Q, as shown in Fig. E.30. The location of point Q relative to the origin O is established with a position vector **r**. Vectors **F** and **r** form a plane π and the moment vector $\mathbf{M_O}$ is perpendicular to this plane.

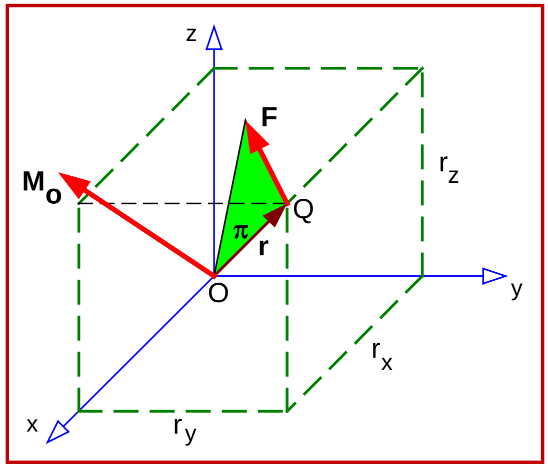


Fig. E.30 A drawing showing the geometric representation of $\mathbf{M_O} = \mathbf{r} \times \mathbf{F}$.

The position vector **r** is determined from the location of point Q as:

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} \qquad \text{(E.43)}$$

where r_x, r_y and r_z are the coordinates of point Q.

Using Eq. (E.18) we write an expression for the force vector as:

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

From Eq. (E.41) it is easy to show that the cross vector products of the Cartesian unit vectors are given by:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0}, & \mathbf{j} \times \mathbf{j} = \mathbf{0}, & \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\,\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\,\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\,\mathbf{j} \end{array} \qquad \text{(E.44)}$$

Combining Eqs. (E.18), (E.42), (E.43) and (E.44), we obtain:

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k} \qquad \text{(E.45)}$$

We may also express the moment in terms of components relative to the coordinate axes as:

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k} \qquad \text{(E.46)}$$

Comparing the results of Eqs. (E.45) with those of Eq. (E.46), yields the equations for the components of the moments about the coordinate axes as:

$$\begin{aligned} M_x &= (r_y F_z - r_z F_y) \\ M_y &= (r_z F_x - r_x F_z) \\ M_z &= (r_x F_y - r_y F_x) \end{aligned} \qquad \text{(E.47)}$$

When employing the vector cross product $\mathbf{r} \times \mathbf{F}$ to determine the moment $\mathbf{M_O}$**,** we simultaneously obtain the relations for the moments M_x, M_y and M_z.

In writing Eq. (E.45), we often employ the determinant given by:

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{E.48}$$

Note that Eqs. (E.45) and (E.46) give the moment $\mathbf{M_O}$ in vector form. The magnitude of the moment, M_O, is given by the square root of the sum of the squares of the moment components as:

$$M_O = [M_x^2 + M_y^2 + M_z^2]^{1/2} \tag{E.49}$$

Let's apply these results to three example problems to demonstrate the use of the vector cross product in the determination of moments due to forces.

EXAMPLE E.26

A force vector **F** with the following components (F_x = 2,000 N, F_y = 2,500 N, and F_z = 3,200 N) is applied to a structure at point Q. If point Q is given by the coordinates x = 3 m, y = 2.5 m, and z = 6 m, determine the moment about the origin O of the coordinate system.

Solution:

The moment is given by Eq. (E.42) as $\mathbf{M_O = r \times F}$. To execute this cross product, we write both **r** and **F** in vector format as:

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} = (3\,\mathbf{i} + 2.5\,\mathbf{j} + 6\,\mathbf{k})\text{ m}$$

$$\mathbf{F} = (2{,}000\,\mathbf{i} + 2{,}500\,\mathbf{j} + 3{,}200\,\mathbf{k})\text{ N} \tag{a}$$

To execute the cross product, let's employ Eq. (E.45) to obtain:

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = (r_y\,F_z - r_z\,F_y)\,\mathbf{i} + (r_z\,F_x - r_x\,F_z)\,\mathbf{j} + (r_x\,F_y - r_y\,F_x)\,\mathbf{k}$$

$$\mathbf{M_O} = [(2.5)(3{,}200) - (6)(2{,}500)]\,\mathbf{i} + [(6)(2{,}000) - (3)(3{,}200)]\,\mathbf{j} + [(3)(2{,}500) - (2.5)(2{,}000)]\,\mathbf{k} \tag{b}$$

$$\mathbf{M_O} = (-\,7{,}000\,\mathbf{i} + 2{,}400\,\mathbf{j} + 2{,}500\,\mathbf{k})\text{ N-m} \tag{c}$$

The magnitude of the moment is given by Eq. (E.49) as:

$$M_O = [M_x^2 + M_y^2 + M_z^2]^{1/2} \tag{d}$$

$$M_O = [(-7.0)^2 + (2.4)^2 + (2.5)^2]^{1/2} = 7.811\text{ kN-m} \tag{e}$$

EXAMPLE E.27

If a structure is loaded at point Q with a force given by:

$$\mathbf{F} = (75\,\mathbf{i} + 120\,\mathbf{j} + 40\,\mathbf{k})\text{ lb}$$

Determine the moment $\mathbf{M_O}$ using the determinant format presented in Eq. (E.48). Note that point Q is located at the position x = 3.0 ft, y = 6.0 ft, and z = 5.0 ft.

Solution:

Let's first write the relation for the position vector **r** as:

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} = (3.0\,\mathbf{i} + 6.0\,\mathbf{j} + 5.0\,\mathbf{k})\text{ ft} \qquad \text{(a)}$$

Recall Eq. (E.48) as:

$$\mathbf{M_O} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \qquad \text{(b)}$$

Substituting into the matrix, the coefficients from the relations for the position and force vectors gives:

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.0 & 6.0 & 5.0 \\ 75 & 120 & 40 \end{vmatrix} \qquad \text{(c)}$$

Solving the determinant gives:

$$\mathbf{M_O} = [(24 - 60)\mathbf{i} + (37.5 - 12)\mathbf{j} + (36 - 45)\mathbf{k}](10)$$

$$\mathbf{M_O} = (-\,360\,\mathbf{i} + 255\,\mathbf{j} - 90\,\mathbf{k})\text{ ft-lb} \qquad \text{(d)}$$

EXAMPLE E.28

The boom of a crane extends from its base at point O to its tip at point Q as indicated in Fig. EXPL.28. A three-dimensional coordinate system has been established in this illustration with the point O at the origin and the point Q defined with coordinates (1,1,5) m. A force with a magnitude of 17 kN is applied by the boom onto a cable that extends from the tip of the boom to point P. Point P is located on the x – y plane with coordinates (3,6,0) m. If the coordinates are expressed in meters, determine the following quantities:

(a) The vector representation of the force **F**.
(b) The components F_x, F_y and F_z.
(c) The moment $\mathbf{M_O}$.
(d) The unit vector specifying the direction of $\mathbf{M_O}$.
(e) The moments about the coordinate axes — M_x, M_y and M_z.
(f) The magnitude of the moment.

(g) The smallest angle between the force and the boom OQ.
(h) The component of F along the boom OQ.

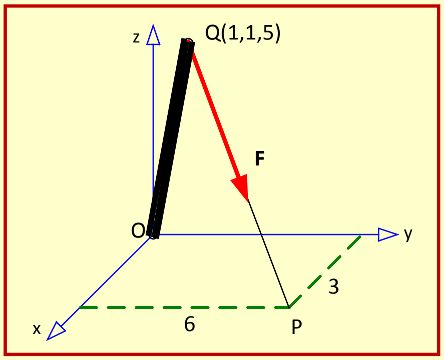


Fig. EXPL.28

Solution:

Let's begin by determining the unit vector that describes the direction of the applied force. Because the force is directed along the cable that originates at point Q and extends through point P, we know its orientation from the coordinates of these two points. The unit vector $\mathbf{u_F}$ can be determined from Eq. (E.29) as:

$$\mathbf{u_F} = \mathbf{r_F}/r_F = [(x_P - x_Q)\mathbf{i} + (y_P - y_Q)\mathbf{j} + (z_P - z_Q)\mathbf{k}]/r_F \tag{a}$$

where r_F is the magnitude of the length of the line from Q to P, given by Eq. (E.28):

$$r_F = [(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2} \tag{b}$$

Substituting the coordinates of points P and Q into Eqs. (a) and (b) gives:

$$\mathbf{u_F} = \frac{(3-1)\mathbf{i} + (6-1)\mathbf{j} + (0-5)\mathbf{k}}{\left[(3-1)^2 + (6-1)^2 + (0-5)^2\right]^{1/2}} = 0.2722\ \mathbf{i} + 0.6804\ \mathbf{j} - 0.6804\ \mathbf{k} \tag{c}$$

The vector representation of the force F is then given by:

$$\mathbf{F} = F\ \mathbf{u_F} = (17)\ \mathbf{u_F} = (4.627\ \mathbf{i} + 11.567\ \mathbf{j} - 11.567\mathbf{k})\text{ kN} \tag{d}$$

The components of the force vector are:

$$F_x = 4.627\text{ kN};\qquad F_y = 11.567\text{ kN};\qquad F_z = -11.567\text{ kN} \tag{e}$$

From the coordinates of point Q, it is clear that the position vector $\mathbf{r}$ is given by:

$$\mathbf{r} = (1\ \mathbf{i} + 1\ \mathbf{j} + 5\ \mathbf{k})\text{ m} \tag{f}$$

The moment is determined from Eq. (E.48) as:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 5 \\ 4.627 & 11.567 & -11.567 \end{vmatrix} \tag{g}$$

Evaluating this determinant yields:

$$\mathbf{M_O} = [(1)(-11.567) - (5)(11.567)]\mathbf{i} + [(5)(4.627) - (1)(-11.567)]\mathbf{j}$$

$$+ [(1)(11.567) - (1)(4.627)]\mathbf{k}$$

$$\mathbf{M_O} = (-\ 69.402\ \mathbf{i} + 34.702\mathbf{j} + 6.940\ \mathbf{k})\ \text{kN-m} \tag{h}$$

The components of this vector relative to the Cartesian coordinates are:

$$M_x = -\ 69.40\ \text{kN-m}; \qquad M_y = 34.70\ \text{kN-m} \qquad M_z = 6.940\ \text{kN-m} \tag{i}$$

The magnitude of the moment M_O is determined from Eq. (E.49) as:

$$M_O = [(-69.40)^2 + (34.70)^2 + (6.940)^2]^{1/2} = 77.90\ \text{kN-m} \tag{j}$$

The unit vector describing the direction of the moment vector M_O is:

$$\mathbf{u}_{Mo} = \mathbf{M_O}/M_O = [-\ 69.40\ \mathbf{i} + 34.70\ \mathbf{j} + 6.940\ \mathbf{k}]/(77.90)$$

$$\mathbf{u}_{Mo} = -0.8909\ \mathbf{i} + 0.4454\ \mathbf{j} + 0.0891\ \mathbf{k} \tag{k}$$

Let's use Eq. (E.39) to determine the angle θ between the force vector **F** and a vector **QO** that extends from point Q to the origin O.

$$\theta = \cos^{-1}\ [\mathbf{F} \bullet \mathbf{QO}/(F)(QO)] \tag{l}$$

Note that **QO** = – **r**; hence, we use Eqs. (d) and (f) to express Eq. (l) as:

$$\theta = \cos^{-1}\left[\frac{(4.627\mathbf{i} + 11.567\mathbf{j} - 11.567\mathbf{k}) \bullet (-1\mathbf{i} - 1\mathbf{j} - 5\mathbf{k})}{(17)(5.196)}\right] = \cos^{-1}\left[0.4714\right] = 61.87° \tag{m}$$

In this calculation, we used the vector **QO** rather than **OQ** to obtain the smallest angle between the force and the direction of the boom. Using **OQ** would have produced an answer of 118.13°, which is the complement of the angle determined.

The component of the force F along the boom OQ is given by:

$$F_{OQ} = F\cos\theta = (17)\cos(118.13°) = -\ 8.015\ \text{kN} \tag{n}$$

Another approach to determine F_{OQ} involves employing Eq. (E.40). Then we write:

$$F_{OQ} = \mathbf{F} \bullet \mathbf{u}_{OQ} = \mathbf{F} \bullet (\mathbf{r_{OQ}}/r_{OQ}) \tag{o}$$

The magnitude r_{OQ} is determined from Eq. (E.28) as:

$$r_{OQ} = [(1)^2 + (1)^2 + (5)^2]^{1/2} = 5.196\ \text{m} \tag{p}$$

Substituting Eqs. (d), (f), and (p) into Eq. (o) gives:

$$F_{OQ} = (4.627\ \mathbf{i} + 11.567\ \mathbf{j} - 11.567\mathbf{k}) \bullet (1\ \mathbf{i} + 1\ \mathbf{j} + 5\ \mathbf{k})/(5.196) = -\ 8.014\ \text{kN} \tag{q}$$

Note that the negative sign on the answer indicates that the boom is under compression and will slightly contract under the action of this force.

E.10 SUMMARY

Both external and internal forces have been described in this appendix. External forces are applied to the structure either by applied loading or by reactions developed at the supports. Internal forces develop within structural members due to the action of the external forces. Section cuts on the structural member are made to visualize the stresses that generate the internal forces.

Forces are vector quantities that require the specification of both magnitude and direction for a complete description. We represent forces with arrows, where its length is proportional to the magnitude and its orientation relative to a suitable coordinate system gives its direction. The line of action of a force is collinear with the force vector. It is possible to slide a force vector along its line of action to a new position without affecting equilibrium of a body.

The process of vector addition and subtraction has been described. We showed that the vector sum $\mathbf{S_v} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ and demonstrated that it could be represented by a triangle with sides of length A, B and S_v. The sine and cosine laws can be employed to determine the magnitude and direction in vector addition or subtraction.

Vector decomposition, the resolution of a single force into two components, was described. Resolution of a force into its two Cartesian components is of particular significance. The equations involved are summarized below.

Classifications of vector systems were discussed with definitions and examples of concurrent and coplanar systems. For concurrent systems, where all of the forces are applied at a common point, it is important to recognize that the equilibrium equation $\Sigma\mathbf{M} = 0$ is satisfied automatically.

Moments were introduced and briefly discussed. Moments, like forces, are vector quantities with both a magnitude and direction. Moments tend to cause bodies, upon which they act, to rotate. The direction of a moment is perpendicular to the plane in which the force and moment arm lie. Moments are considered **positive** when they tend to produce a **counterclockwise** rotation of the body upon which they act.

Space forces were described with force components in three directions (x, y and z). Coordinate direction angles were defined, to provide the orientation of the force vectors in the three-dimensional coordinate system. Unit vectors **i**, **j** and **k** directed along the Cartesian axes were defined. It was shown that the direction of space forces could be expressed in terms of a unit vector $\mathbf{u_F}$. A position vector **r** was introduced to locate the point of application of a force in space.

Finally, vector algebra was introduced and the three dimensional vector representations of forces, moments and unit vectors were given. These representations permit us to show separately the direction and magnitude of a vector quantity. The vector dot product was described together with two useful applications in analysis of forces acting on structures. The vector cross product was also defined. It is used extensively in determining the moments produced by forces acting on three-dimensional structures. Examples demonstrating the application of the vector dot and cross products were provided.

Key relations you will use on many occasions in the analysis of structures that were introduced in this appendix are summarized below.

$$\boxed{P_{int} = \int \sigma \, dA \qquad \text{(E.1)}}$$

$$\boxed{\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \qquad \text{(E.4)}}$$

$$\boxed{C = \sqrt{A^2 + B^2 - 2AB\cos(c)} \qquad \text{(E.5)}}$$

For two dimensional forces and moments:

$$\mathbf{F} = \mathbf{F_x} + \mathbf{F_y} \qquad \text{(E.7)}$$

$$F_x = F \cos \theta \text{ and } F_y = F \sin \theta \qquad \text{(E.8)}$$

$$F = [F_x^{\,2} + F_y^{\,2}]^{1/2} \qquad \text{(E.9)}$$

$$M_o = (F)(d) \qquad \text{(E.30)}$$

We indicate the direction of the space forces with coordinate direction angles defined by:

$$\cos \alpha = \frac{F_x}{F} \quad \cos \beta = \frac{F_y}{F} \quad \cos \gamma = \frac{F_z}{F} \qquad \text{(E.14)}$$

Alternatively the direction may be specified by unit vectors directed along the line of action of the space force.

$$\mathbf{u}_F = \mathbf{F}/F \qquad \text{(E.19)}$$

For space forces (three-dimensional) and moments it is often easier to visualize the vectors and more efficient to perform the analysis by employing vector mechanics. The relations frequently used for three-dimensional analyses include vector representation of forces, moments and position. Also, extensive utilization of the dot and cross products facilitates the computation required.

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \qquad \text{(E.18)}$$

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} \qquad \text{(E.26)}$$

$$\cos \alpha = (x_B - x_A)/r \quad \cos \beta = (y_B - y_A)/r \quad \cos \gamma = (z_B - z_A)/r \qquad \text{(E.27)}$$

$$r = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \qquad \text{(E.28)}$$

$$\mathbf{u}_r = \mathbf{r}/r = (r_x/r)\mathbf{i} + (r_y/r)\mathbf{j} + (r_z/r)\mathbf{k} \qquad \text{(E.29)}$$

$$\mathbf{A} \bullet \mathbf{B} = (A)(B) \cos \theta \qquad \text{(E.36)}$$

$$\mathbf{A} \bullet \mathbf{B} = A_x\,B_x + A_y\,B_y + A_z\,B_z \qquad \text{(E.38)}$$

$$\theta = \cos^{-1} [(\mathbf{A} \bullet \mathbf{B})/(A)(B)] \qquad \text{(E.39)}$$

$$A_u = \mathbf{A} \bullet \mathbf{u} \qquad \text{(E.40)}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = [(A)(B)\sin\theta]\mathbf{u} \qquad \text{(E.41)}$$

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} \qquad \text{(E.42)}$$

$$\mathbf{r} = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k} \qquad \text{(E.43)}$$

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k} \qquad \text{(E.45)}$$

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k} \qquad \text{(E.46)}$$

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \qquad \text{(E.48)}$$

$$M_O = [M_x^2 + M_y^2 + M_z^2]^{1/2} \qquad \text{(E.49)}$$

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Unit Conversion Factors

Quantity	U. S. Customary	SI Equivalent
Acceleration	ft/s^2 in/s^2	$0.3048\ m/s^2$ $0.0254\ m/s^2$
Area	ft^2 in^2	$0.0929\ m^2$ $645.2\ mm^2$
Distributed Load	lb/ft lb/in.	14.59 N/m 0.1751 N/mm
Energy	ft-lb	1.356J
Force	kip = 1000 lb lb	4.448 kN 4.448N
Impulse	lb-s	4.448 N-s
Length	ft in mi	0.3048 m 25.40 mm 1.609 km
Mass	lb mass slug ton mass	0.4536 kg 14.59 kg 907.2 kg
Moments or Torque	ft-lb in-lb	1.356 N-m 0.1130 N-m
Area Moment of Inertia	in^4	$0.4162 \times 10^6\ mm^4$
Power	ft-lb/s hp	1.356 W 745.7 W
Stress and Pressure	lb/ft^2 lb/in^2 (psi) ksi = 1000 psi	47.88 Pa 6.895 kPa 6.895 MPa
Velocity	ft/s in/s mi/h (mph)	0.3048 m/s 0.0254 m/s 0.4470 m/s
Volume	ft^3 in^3 gal	$0.02832\ m^3$ $16.39\ cm^3$ 3.785 L
Work or Energy	ft-lb	1.356 J

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